Analysis of Positioning Uncertainty in Cooperative Localization and Target Tracking (CLATT)

Faraz M. Mirzaei , Anastasios I. Mourikis and Stergios I. Roumeliotis
{faraz|mourikis|stergios}@cs.umn.edu

Dept. of Computer Science & Engineering
University of Minnesota
Minneapolis, MN 55455

MARS LAB

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Robotic Systems Laboratory

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Dept. of Computer Science & Engineering
University of Minnesota
4-192 EE/CS Building
200 Union St. S.E.
Minneapolis, MN 55455
Tel: (612) 625-2217
Fax: (612) 625-0572
URL: http://www.cs.umn.edu/~faraz

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Abstract

In this report, we study the positioning accuracy of Cooperative Localization and Target Tracking (CLATT) in a network of mobile robots, and derive analytical upper bounds for the positioning uncertainty. The obtained bounds provide a description for the asymptotic positioning performance of the robots and the targets as a function of the sensor characteristics, and the structure of the graph of relative position measurements. By employing an Extended Kalman Filter (EKF) formulation for data fusion, two expressions are derived for the asymptotic positioning performance. One expression provides the guaranteed worst-case positioning accuracy, whereas the other determines an upper bound on the *expected* position estimates' covariance. We investigate the effects of jointly estimating the targets' and the robots' position, and demonstrate that it results in *better* accuracy for the robots' position estimates.

1 Introduction

The idea of employing sensor networks for target tracking has recently become attractive, as the result of advances in the manufacturing of low-cost communication and sensing devices. When multiple nodes of a sensor network measure the distance and bearing to a target of interest, the acquired data can be processed (either at a central fusion center or in a distributed fashion) in order to estimate the target's position. When instead of *static*, *mobile* sensors are employed, a larger area can be covered without the need to increase the number of nodes in the sensing network [1]. Moreover, the configuration of the sensors can be changed so as to adapt to the motion of the target. For example, a team of robots can actively pursue a target to prevent the target's escape from the visibility range of sensors [2].

When a team of robots is employed to track a number of targets, the position of the robots (*Localization*) and the position of the targets (*Tracking*) need to be concurrently estimated. In this report, we study the problem of Cooperative Localization and Target Tracking (CLATT) in scenarios where teams of, possibly heterogeneous, mobile robots track the position of multiple targets. One of the main results of this report is a proof that jointly estimating the position of the robots and targets results in *better* accuracy for the robots' position estimates, compared to when the robots localize ignoring the targets. Intuitively, this can be justified by considering that the robots are implicitly performing observations of their relative positions by measuring the range and bearing to the same targets.

Another significant contribution of this report is to examine *the robots' performance on average and in the worst case*. This is a common question that is always addressed before any investment in system development is made. In this work, the term performance refers to the accuracy of position estimation and is assessed by the covariance of the position estimates for the robots and targets. *Analytical* upper bounds for the uncertainty of the robots' and the targets' localization are presented that are functions of the sensors' characteristics and of the structure of the sensing graph that connects the robots and targets. Furthermore, it is shown how a priori information about the distribution of the positions of the robots and the targets can be utilized to derive an upper bound for the *expected* value of the position estimates' covariance. The developed upper bounds can be employed in order to *predict* the position accuracy attained in a certain tracking application, and thus can facilitate the task of sensor selection, so as to meet the requirements of the application.

In this report, we consider the case where: (i) all robots are equipped with proprioceptive sensors that measure velocity, and are provided with orientation estimates of bounded uncertainty, (ii) one of the robots is equipped with a GPS receiver, enabling it to obtain absolute position estimates, and (iii) the robots are capable of measuring the relative positions of other robots and targets. These robot-to-robot and robot-to-target measurements are presented by the *Relative Position Measurement Graph* (PRMG). This is a connected and directed graph, whose vertices represent the robots and targets and edges represent the relative position measurements between them.

2 Related Work

The problem of target tracking using multi-sensor networks has been the subject of extensive research in recent years [3, 4, 5]. Most researchers address the problem of tracking with a network of *static* sensors, nevertheless, several approaches have been proposed for target tracking with mobile robots. For example, Parker [6] has developed a control strategy for multi-robot teams to minimize the total time in which the target can evade being observed by the robots. Several region-based approaches to target tracking by mobile robots have been developed by Jung and Sukhatme [1, 7]. A hierarchical algorithm for localization and tracking using directional sensors is presented in [8]. Stroupe et al. [9] propose a distributed action-selection algorithm that can be employed in order to optimize the robots' trajectories, with respect to the targets' position covariance. Despite their importance for practical applications, none of the aforementioned approaches address the problem of determining bounds on the performance (accuracy) of the robots' and targets' localization.

A number of approaches aimed at providing a description of the localization accuracy during target tracking have been developed for the case in which the sensors remain *static*. In [10], an optimal approach for fusing target tracking data is developed, and its performance is evaluated using Monte Carlo simulations. An analytical performance evaluation of this method is also provided in [11]. Zhang *et al.* [12] have studied the *Cramer-Rao Lower Bound* (CRLB) of the covariance of a target's position estimates. The CRLB for tracking manoeuvering targets is presented in [13]. In [14], the performance of a wireless sensor network in the presence of communication delays and false alarms is analyzed and compared to the CRLB. All aforementioned approaches focus on static sensor networks exclusively, and the results cannot be readily extended to networks of mobile sensors. Additionally, although the CRLB is especially useful for the evaluation of suboptimal target tracking algorithms, it cannot be employed to determine the *worst-case* performance of tracking, which is of interest before the deployment of a system in any application.

The main contribution of the work presented in this report is the characterization of the steady-state accuracy of the position estimates in Cooperative Localization and Target Tracking (CLATT). This is achieved by deriving *analytical upper bounds* of the steady-state covariance matrix of the position estimate, for the worst-case scenario as well as for the *average* case, i.e., the case where a probabilistic description of the targets' and robots' positions is known in advance. Moreover, when the process noise in the target motion model is infinite (i.e., no prior information of the target's motion model is available), a study of the localization accuracy provides a worst-case performance bound over all possible motion models that employ prior information. This analysis also demonstrates that the robot's position estimates are always *better* when, in addition to robot-to-robot measurements, the robots also process robot-to-target measurements.

3 Problem Formulation

3.1 Position propagation

The discrete-time kinematic equations for the i-th robot are

$$x_{r_i}(k+1) = x_{r_i}(k) + V_i(k)\delta t \cos(\phi_i(k)) \quad (1)$$

$$y_{r_i}(k+1) = y_{r_i}(k) + V_i(k)\delta t \sin(\phi_i(k)) \quad (2)$$

where $V_i(k)$ denotes the robot's translational velocity at time k and δt is the sampling period. These equations imply that the robots are moving in a 2D plane. In the Kalman filter framework, the estimates of the robot's position are propagated using the measurements of the robot's velocity, $V_{m_i}(k)$, and the estimates of the robot's orientation, $\hat{\phi}_i(k)$:

$$\begin{aligned}\hat{x}_{r_{i_{k+1|k}}} &= \hat{x}_{r_{i_{k|k}}} + V_{m_i}(k)\delta t\cos(\hat{\phi}_i(k))\\ \hat{y}_{r_{i_{k+1|k}}} &= \hat{y}_{r_{i_{k|k}}} + V_{m_i}(k)\delta t\sin(\hat{\phi}_i(k))\end{aligned}$$

Clearly, these equations are time varying and nonlinear due to the dependence on the robot's orientation. By linearizing Eqs. (1) and (2), the error propagation equation for the robot's position is readily derived:

$$\begin{aligned}\begin{bmatrix}\widetilde{x}_{r_{i_{k+1|k}}}\\ \widetilde{y}_{r_{i_{k+1|k}}}\end{bmatrix} &= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\widetilde{x}_{r_{i_{k|k}}}\\ \widetilde{y}_{r_{i_{k|k}}}\end{bmatrix} + \begin{bmatrix}\delta t\cos(\hat{\phi}_i(k)) & -V_{m_i}(k)\delta t\sin(\hat{\phi}_i(k))\\ \delta t\sin(\hat{\phi}_i(k)) & V_{m_i}(k)\delta t\cos(\hat{\phi}_i(k))\end{bmatrix}\begin{bmatrix}w_{V_i}(k)\\ \widetilde{\phi}_i(k)\end{bmatrix}\\ \Leftrightarrow \widetilde{X}_{r_{i_{k+1|k}}} &= \quad I_{2\times 2} \quad \widetilde{X}_{r_{i_{k|k}}} \quad + \quad G_{r_i}(k) \quad W_i(k)\end{aligned} \tag{3}$$

where[1] $w_{V_i}(k)$ is a zero-mean white Gaussian noise sequence of variance $\sigma^2_{V_i}$, affecting the velocity measurements and $\widetilde{\phi}_i(k)$ is the error in the robot's orientation estimate at time k. This is modeled as a zero-mean white Gaussian noise sequence of variance $\sigma^2_{\phi_i}$.

From Eq. (3), we deduce that the covariance matrix of the system noise affecting the i-th robot is:

$$\begin{aligned}Q_{r_i}(k) &= E\{G_{r_i}(k)W_i(k)W_i^T(k)G_{r_i}^T(k)\}\\ &= G_{r_i}(k)E\{W_i(k)W_i^T(k)\}G_{r_i}^T(k)\\ &= \begin{bmatrix}\delta t\cos(\hat{\phi}_i(k)) & -V_{m_i}(k)\delta t\sin(\hat{\phi}_i(k))\\ \delta t\sin(\hat{\phi}_i(k)) & V_{m_i}(k)\delta t\cos(\hat{\phi}_i(k))\end{bmatrix}\begin{bmatrix}\sigma^2_{V_i} & 0\\ 0 & \sigma^2_{\phi_i}\end{bmatrix}\begin{bmatrix}\delta t\cos(\hat{\phi}_i(k)) & -V_{m_i}(k)\delta t\sin(\hat{\phi}_i(k))\\ \delta t\sin(\hat{\phi}_i(k)) & V_{m_i}(k)\delta t\cos(\hat{\phi}_i(k))\end{bmatrix}^T\\ &= \begin{bmatrix}\cos(\hat{\phi}_i(k)) & -\sin(\hat{\phi}_i(k))\\ \sin(\hat{\phi}_i(k)) & \cos(\hat{\phi}_i(k))\end{bmatrix}\begin{bmatrix}\delta t^2\sigma^2_{V_i} & 0\\ 0 & \delta t^2 V^2_{m_i}(k)\sigma^2_{\phi_i}\end{bmatrix}\begin{bmatrix}\cos(\hat{\phi}_i(k)) & -\sin(\hat{\phi}_i(k))\\ \sin(\hat{\phi}_i(k)) & \cos(\hat{\phi}_i(k))\end{bmatrix}^T\\ &= C(\hat{\phi}_i(k))\begin{bmatrix}\delta t^2\sigma^2_{V_i} & 0\\ 0 & \delta t^2 V^2_m(k)\sigma^2_{\phi_i}\end{bmatrix}C^T(\hat{\phi}_i(k))\end{aligned} \tag{4}$$

where $C(\hat{\phi}_i)$ denotes the 2×2 rotation matrix associated with $\hat{\phi}_i$.

A zero velocity model is used to model the targets' 2D motion [15], hence, the state propagation equations for the i-th target are

$$X_{T_{i_{k+1|k}}} = X_{T_{i_{k|k}}} + \delta t W_{T_i}(k)$$

where $W_{T_i} = [w_{T_{x_i}} \;\; w_{T_{y_i}}]^T$ is the noise process, introduced in the target's motion model to express the uncertainty in the actual motion of the target.

The error propagation equation is given by:

$$\begin{aligned}\begin{bmatrix}\widetilde{x}_{T_{i_{k+1|k}}}\\ \widetilde{y}_{T_{i_{k+1|k}}}\end{bmatrix} &= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\widetilde{x}_{T_{i_{k|k}}}\\ \widetilde{y}_{T_{i_{k|k}}}\end{bmatrix} + \delta t\begin{bmatrix}w_{T_{i_x}}(k)\\ w_{T_{i_y}}(k)\end{bmatrix}\\ \Leftrightarrow \widetilde{X}_{Tk+1|k} &= \quad I_{2\times 2} \quad \widetilde{X}_{Tk|k} \quad + \quad \delta t\, W_T(k)\end{aligned} \tag{5}$$

and the covariance of the system noise of the target is

$$Q_T = E\left\{\delta t^2 W_T(k)W_T^T(k)\right\} = \delta t^2\begin{bmatrix}\sigma^2_{T_{ix}} & 0\\ 0 & \sigma^2_{T_{iy}}\end{bmatrix} = \delta t^2\sigma^2_T I_{2\times 2} = Q_T \tag{6}$$

where σ^2_T is the variance of the targets' system noise, assumed to be homogenous along all three axes, and identical for all targets. The state vector for the entire system is defined as the stacked vector comprising of the positions of the robots and the targets, i.e.,

$$X = \begin{bmatrix} X^T_{r_1} & \cdots & X^T_{r_M} & X^T_{T_1} & \cdots & X^T_{T_N}\end{bmatrix}^T$$

[1]Throughout this report, $\mathbf{0}_{m\times n}$ denotes the $m\times n$ matrix of zeros, $\mathbf{1}_{m\times n}$ denotes the $m\times n$ matrix of ones, and $I_{n\times n}$ denotes the $n\times n$ identity matrix.

Hence, the state transition matrix for the entire system at time-step k is $\mathbf{\Phi}_k = I_{2M+2N}$, and the covariance matrix of the system noise is:

$$\mathbf{Q}(k) \quad = \quad \begin{bmatrix} \mathbf{Q}_r(k) & \mathbf{0}_{2M\times 2N} \\ \mathbf{0}_{2N\times 2M} & \mathbf{Q}_T \end{bmatrix} \tag{7}$$

where $\mathbf{Q}_r(k) = \mathbf{Diag}(Q_{r_i}(k))$, and $\mathbf{Q}_T = I_N \otimes (Q_T)$ are block diagonal matrices describing the system noise covariance for the robots and targets, respectively.

The equation for propagating the covariance matrix of the state error is written as:

$$\mathbf{P}_{k+1|k} = \mathbf{P}_{k|k} + \mathbf{Q}(k) \tag{8}$$

where $\mathbf{P}_{k+1|k} = E\{\widetilde{X}_{k+1|k}\widetilde{X}^T_{k+1|k}\}$ and $\mathbf{P}_{k|k} = E\{\widetilde{X}_{k|k}\widetilde{X}^T_{k|k}\}$ are the covariance of the error in the estimate of $X(k+1)$ and $X(k)$ respectively, after measurements up to time k have been processed.

3.2 Measurement Model

At every time step, the robots perform robot-to-robot and robot-to-target relative position measurements. Also one of the robots, say the first one is equipped with GPS. The relative position measurement between robots r_i and r_m is given by:

$$z_{r_i r_m} = C^T(\phi_i)\left(X_{r_m} - X_{r_i}\right) + n_{z_{r_i r_m}} \tag{9}$$

where r_i (r_m) is the observing (observed) robot, and $n_{z_{r_i r_m}}$ is the noise affecting this measurement. Similarly, the measurement of the relative position between r_i and the target is given by:

$$z_{r_i T} = C^T(\phi_i)\left(X_T - X_{r_i}\right) + n_{z_{r_i T}} \tag{10}$$

The similarity of the preceding two measurement equations allows us to treat both types of measurements in a uniform manner. We denote by S_{ij} the subject of the j-th measurement performed by robot i, i.e.,

$$S_{ij} \in \{r_1, r_2, \ldots, r_M, T_1, T_2, \ldots, T_N\} \setminus \{r_i\}$$

Thus, the general form of the relative position measurement equation is:

$$z_{ij} = C^T(\phi_i)\left(X_{S_{ij}} - X_{r_i}\right) + n_{z_{ij}} \tag{11}$$

Assuming that the i-th robot performs M_i relative position measurements, the index j assumes integer values in the range $[1, M_i]$ to describe these measurements. By linearizing the last expression, the measurement error equation is obtained:

$$\begin{aligned}
\widetilde{z}_{ij}(k+1) &= z_{ij}(k+1) - \hat{z}_{ij}(k+1) \\
&= C^T(\hat{\phi}_i(k+1))\left(\widetilde{X}_{S_{ij}\ k+1|k} - \widetilde{X}_{r_i\ k+1|k}\right) - C^T(\hat{\phi}_i(k+1))J\left(\hat{X}_{S_{ij}\ k+1|k} - \hat{X}_{r_i\ k+1|k}\right)\widetilde{\phi}_i(k+1) + n_{z_{ij}}(k+1) \\
&= C^T(\hat{\phi}_i(k+1))\begin{bmatrix} 0_{2\times2} & \cdots & \underbrace{-I_{2\times2}}_{r_i} & \cdots & \underbrace{I_{2\times2}}_{S_{ij}} & \cdots & 0_{2\times2} \end{bmatrix}\begin{bmatrix} \vdots \\ \widetilde{X}_{r_i} \\ \vdots \\ \widetilde{X}_{S_{ij}} \\ \vdots \end{bmatrix}_{k+1|k} \\
&+ \begin{bmatrix} I_{2\times2} & -C^T(\hat{\phi}_i(k+1))J\widehat{\Delta p}_{ij\ k+1|k} \end{bmatrix}\begin{bmatrix} n_{z_{ij}}(k+1) \\ \widetilde{\phi}_i(k+1) \end{bmatrix} \\
&= H_{ij}(k+1)\widetilde{X}_{k+1|k} + \Gamma_{ij}(k+1)n_{ij}(k+1)
\end{aligned} \tag{12}$$

where

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \widehat{\Delta p}_{ij_{k+1|k}} = \hat{X}_{S_{ij}\ k+1|k} - \hat{X}_{r_i\ k+1|k}$$

and we note that the measurement matrix for this relative position measurement can be written as

$$H_{ij}(k+1) = C^T(\hat{\phi}_i(k+1)) \begin{bmatrix} 0_{2\times 2} & \dots & \underbrace{-I_{2\times 2}}_{r_i} & \dots & \underbrace{I_{2\times 2}}_{S_{ij}} & \dots & 0_{2\times 2} \end{bmatrix} = C^T(\hat{\phi}_i(k+1))H_{o_{ij}} \tag{13}$$

At each time instant robot i records M_i relative position measurements, described by the measurement matrix $\boldsymbol{H}_i(k+1)$, i.e., a matrix whose block rows are $H_{ij}(k+1)$, $j = 1 \dots M_i$, i.e.:

$$\boldsymbol{H}_i(k+1) = \begin{bmatrix} H_{i1} \\ H_{i2} \\ \vdots \\ H_{iM_i} \end{bmatrix} = \begin{bmatrix} C^T(\hat{\phi}_i(k+1))H_{o_{i1}} \\ C^T(\hat{\phi}_i(k+1))H_{o_{i2}} \\ \vdots \\ C^T(\hat{\phi}_i(k+1))H_{o_{iM_i}} \end{bmatrix} = \boldsymbol{\Xi}^T_{\hat{\phi}_i}(k+1)\boldsymbol{H}_{o_i} \tag{14}$$

in the last expression $\boldsymbol{H}_{o_i}$ is a constant matrix whose block rows are $H_{o_{ij}}$, $j = 1 \dots M_i$, and $\boldsymbol{\Xi}_{\hat{\phi}_i}(k+1) = I_{M_i \times M_i} \otimes C(\hat{\phi}_i(k+1))$, with $\otimes$ denoting the Kronecker matrix product.

In addition to measuring relative position measurements, one robot receives measurements of its position in the global coordinate frame. Assuming that robot r_1 is equipped with a GPS, the associated measurement matrix is:

$$\boldsymbol{H}_{GPS} = \left[I_{2\times 2}\ ,\ \mathbf{0}_{2\times(2M+2N-2)}\right]$$

The measurement matrix $\mathbf{H}(k+1)$ describing all the measurements that are performed time step $k+1$ is a matrix with block rows $\boldsymbol{H}_i(k+1)$, $i = 1 \dots M$, i.e.,

$$\mathbf{H}_R(k+1) = \begin{bmatrix} \boldsymbol{\Xi}^T_{\hat{\phi}_0}\boldsymbol{H}_{o_0} \\ \boldsymbol{\Xi}^T_{\hat{\phi}_1}(k+1)\boldsymbol{H}_{o_1} \\ \boldsymbol{\Xi}^T_{\hat{\phi}_2}(k+1)\boldsymbol{H}_{o_2} \\ \vdots \\ \boldsymbol{\Xi}^T_{\hat{\phi}_M}(k+1)\boldsymbol{H}_{o_M} \end{bmatrix} = \mathbf{Diag}\left(\boldsymbol{\Xi}^T_{\hat{\phi}_i}(k+1)\right) \begin{bmatrix} \boldsymbol{H}_{o_0} \\ \boldsymbol{H}_{o_1} \\ \boldsymbol{H}_{o_2} \\ \vdots \\ \boldsymbol{H}_{o_M} \end{bmatrix} = \boldsymbol{\Xi}^T(k+1)\mathbf{H}_o \tag{15}$$

where $\boldsymbol{\Xi}(k+1) = \mathbf{Diag}\left(\boldsymbol{\Xi}_{\hat{\phi}_i}(k+1)\right)$ is a block diagonal matrix with block elements $\boldsymbol{\Xi}_{\hat{\phi}_i}(k+1)$, for $i = 0 \dots M$, $\boldsymbol{\Xi}_{\hat{\phi}_0}(k+1)$ is identity rotation matrix, and $\mathbf{H}_o$ is a matrix with block rows $\boldsymbol{H}_{o_i}$, $i = 0 \dots M$ while $\boldsymbol{H}_{o_0} = \boldsymbol{H}_{GPS}$.

- **Measurement Error**

The covariance for the error of the j-th measurement of robot i is given by

$$\begin{aligned} {}^iR_{jj}(k+1) &= \Gamma_{ij}(k+1)E\{n_{ij}(k+1)n^T_{ij}(k+1)\}\Gamma^T_{ij}(k+1) \\ &= R_{z_{ij}}(k+1) + R_{\tilde{\phi}_{ij}}(k+1) \end{aligned} \tag{16}$$

This expression encapsulates all sources of noise and uncertainty that contribute to the measurement error $\widetilde{z}_{ij}(k+1)$. More specifically, $R_{z_{ij}}(k+1)$ is the covariance of the noise $n_{z_{ij}}(k+1)$ in the recorded relative position measurement $z_{ij}(k+1)$ and $R_{\tilde{\phi}_{ij}}(k+1)$ is the additional covariance term due to the error $\tilde{\phi}_i(k+1)$ in the orientation estimate of the measuring robot. The latter is given by:

$$\begin{aligned} R_{\tilde{\phi}_{ij}}(k+1) &= C^T(\hat{\phi}_i(k+1))J\widehat{\Delta p}_{ij_{k+1|k}}E\{\tilde{\phi}_i^{\ 2}\}\widehat{\Delta p}^T_{ij_{k+1|k}}J^TC(\hat{\phi}_i(k+1)) \\ &= \sigma^2_{\phi_i}C^T(\hat{\phi}_i(k+1))J\widehat{\Delta p}_{ij_{k+1|k}}\widehat{\Delta p}^T_{ij_{k+1|k}}J^TC(\hat{\phi}_i(k+1)) \end{aligned} \tag{17}$$

From this expression we conclude that the uncertainty $\sigma^2_{\hat{\phi}_i}$ in the orientation estimate $\hat{\phi}_i(k+1)$ of the robot is amplified by the distance between the robot and corresponding target. Each relative position measurement is comprised of the range ρ_{ij} and bearing θ_{ij} of the target, expressed in the measuring robot's local coordinate frame, i.e.,

$$z_{ij}(k+1) = \begin{bmatrix} \rho_{ij}(k+1)\cos\gamma_{ij}(k+1)\cos\theta_{ij}(k+1) \\ \rho_{ij}(k+1)\cos\gamma_{ij}(k+1)\sin\theta_{ij}(k+1) \end{bmatrix} + n_{z_{ij}}(k+1) \tag{18}$$

By linearizing, the noise in this measurement can be expressed as:

$$n_{z_{ij}}(k+1) \simeq \begin{bmatrix} \cos\hat{\gamma}_{ij}\cos\hat{\theta}_{ij} & -\hat{\rho}_{ij}\sin\hat{\gamma}_{ij}\cos\hat{\theta}_{ij} \\ \cos\hat{\gamma}_{ij}\sin\hat{\theta}_{ij} & -\hat{\rho}_{ij}\sin\hat{\gamma}_{ij}\sin\hat{\theta}_{ij} \end{bmatrix} \begin{bmatrix} n_{\rho_{ij}}(k+1) \\ n_{\gamma_{ij}}(k+1) \end{bmatrix}$$

where $n_{\rho_{ij}}$ is the error in the range measurement, $n_{\theta_{ij}}$ is the error in the bearing measurement, assumed to be independent white zero-mean Gaussian sequences, and

$$\hat{\rho}^2_{ij} = \widehat{\Delta p}^T_{ij_{k+1|k}} \widehat{\Delta p}_{ij_{k+1|k}}$$
$$\hat{\theta}_{ij} = \text{Atan2}(\widehat{\Delta y}_{ij_{k+1|k}}, \widehat{\Delta x}_{ij_{k+1|k}}) - \hat{\phi}_i(k+1)$$

are the estimates of the range, bearing and planar range of the targets, expressed with respect to the robot's coordinate frame. At this point we note that

$$\begin{aligned} & C(\hat{\phi}_i(k+1)) n_{z_{ij}}(k+1) = \\ & \begin{bmatrix} \cos\hat{\phi}_i(k+1) & -\sin\hat{\phi}_i(k+1) \\ \sin\hat{\phi}_i(k+1) & \cos\hat{\phi}_i(k+1) \end{bmatrix} \begin{bmatrix} \cos\hat{\theta}_{ij} & -\hat{\rho}_{ij}\sin\hat{\theta}_{ij} \\ \sin\hat{\theta}_{ij} & \hat{\rho}_{ij}\cos\hat{\theta}_{ij} \end{bmatrix} \begin{bmatrix} n_{\rho_{ij}}(k+1) \\ n_{\theta_{ij}}(k+1) \end{bmatrix} = \\ & \begin{bmatrix} \cos(\hat{\phi}_i(k+1)+\hat{\theta}_{ij}) & -\hat{\rho}_{ij}\sin(\hat{\phi}_i(k+1)+\hat{\theta}_{ij}) \\ \sin(\hat{\phi}_i(k+1)+\hat{\theta}_{ij}) & \hat{\rho}_{ij}\cos(\hat{\phi}_i(k+1)+\hat{\theta}_{ij}) \end{bmatrix} \begin{bmatrix} n_{\rho_{ij}}(k+1) \\ n_{\theta_{ij}}(k+1) \end{bmatrix} = \\ & \begin{bmatrix} \frac{1}{\hat{\rho}_{ij}}\widehat{\Delta p}_{ij} & J\widehat{\Delta p}_{ij} \end{bmatrix} \begin{bmatrix} n_{\rho_{ij}}(k+1) \\ n_{\theta_{ij}}(k+1) \end{bmatrix} \end{aligned} \tag{19}$$

and therefore considering $C^T(\phi)C(\phi) = I_{2\times2}$, the quantity $R_{z_{ij}}(k+1)$ can be written as:

$$\begin{aligned} R_{z_{ij}}(k+1) &= E\{n_{z_{ij}}(k+1)n^T_{z_{ij}}(k+1)\} \\ &= C^T(\hat{\phi}_i(k+1)) \begin{bmatrix} \frac{1}{\hat{\rho}_{ij}}\widehat{\Delta p}_{ij} \\ J\widehat{\Delta p}_{ij} \end{bmatrix}^T E\left\{ \begin{bmatrix} n_{\rho_{ij}} \\ n_{\theta_{ij}} \end{bmatrix} \begin{bmatrix} n_{\rho_{ij}} \\ n_{\theta_{ij}} \end{bmatrix}^T \right\} \begin{bmatrix} \frac{1}{\hat{\rho}_{ij}}\widehat{\Delta p}_{ij} \\ J\widehat{\Delta p}_{ij} \end{bmatrix} C(\hat{\phi}_i(k+1)) \\ &= C^T(\hat{\phi}_i(k+1)) \begin{bmatrix} \frac{1}{\hat{\rho}_{ij}}\widehat{\Delta p}_{ij} \\ J\widehat{\Delta p}_{ij} \end{bmatrix}^T \begin{bmatrix} \sigma^2_{\rho_i} & 0 \\ 0 & \sigma^2_{\theta_i} \end{bmatrix} \begin{bmatrix} \frac{1}{\hat{\rho}_{ij}}\widehat{\Delta p}_{ij} \\ J\widehat{\Delta p}_{ij} \end{bmatrix} C(\hat{\phi}_i(k+1)) \\ &= C^T(\hat{\phi}_i(k+1)) \left(\frac{\sigma^2_{\rho_i}}{\hat{\rho}^2_{ij}} \widehat{\Delta p}_{ij}\widehat{\Delta p}^T_{ij} + \sigma^2_{\theta_i} J\widehat{\Delta p}_{ij}\widehat{\Delta p}^T_{ij}J^T \right) C(\hat{\phi}_i(k+1)) \\ &= C^T(\hat{\phi}_i(k+1)) \left(\frac{\sigma^2_{\rho_i}}{\hat{\rho}^2_{ij}} \left(\hat{\rho}^2_{ij} I_{2\times2} - J\widehat{\Delta p}_{ij}\widehat{\Delta p}^T_{ij}J^T \right) + \sigma^2_{\theta_i} J\widehat{\Delta p}_{ij}\widehat{\Delta p}^T_{ij}J^T \right) C(\hat{\phi}_i(k+1)) \\ &= C^T(\hat{\phi}_i(k+1)) \left(\sigma^2_{\rho_i} I_{2\times2} + \left(\sigma^2_{\theta_i} - \frac{\sigma^2_{\rho_i}}{\hat{\rho}^2_{ij}} \right) J\widehat{\Delta p}_{ij}\widehat{\Delta p}^T_{ij}J^T \right) C(\hat{\phi}_i(k+1)) \end{aligned} \tag{20}$$

where the variance of the noise in the distance and bearing measurements is given by

$$\sigma^2_{\rho_i} = E\{n^2_{\rho_i}\} \ , \ \ \sigma^2_{\theta_i} = E\{n^2_{\theta_i}\}$$

respectively. Due to the existence of the error component attributed to $\tilde{\phi}_i(k+1)$, the exteroceptive measurements that each robot performs at a given time instant are correlated. The matrix of correlation between the errors in the measurements $z_{ij}(k+1)$ and $z_{i\ell}(k+1)$ is

$${}^iR_{j\ell}(k+1) = \Gamma_{ij}(k) E\{n_{ij}(k+1)n^T_{i\ell}(k+1)\}\Gamma^T_{i\ell}(k)$$

$$= \quad \sigma_{\phi_i}^2 C^T(\hat{\phi}_i(k+1)) J\widehat{\Delta p}_{ij_{k+1|k}} \widehat{\Delta p}_{i\ell_{k+1|k}}^T J^T C(\hat{\phi}_i(k+1)) \tag{21}$$

The covariance matrix of all the measurements performed by robot i at the time instant $k+1$ can now be computed. This is a block matrix whose mn-th 2×2 submatrix element is ${}^iR_{mn}$, for $m, n = 1 \dots M_i$. Using the results of Eqs. (17), (20), and (21), this matrix can be written as

$$\boldsymbol{R}_i(k+1) = \boldsymbol{\Xi}_{\hat{\phi}_i}^T(k+1)\boldsymbol{R}_{o_i}(k+1)\boldsymbol{\Xi}_{\hat{\phi}_i}(k+1) \tag{22}$$

where

$$\boldsymbol{R}_{o_i}(k+1) =$$

$$\begin{bmatrix} \sigma_{\rho_i}^2 I_{2\times 2} + \left(\sigma_{\phi_i}^2 + \sigma_{\theta_i}^2 - \frac{\sigma_{\rho_i}^2}{\hat{\rho}_{i1}^2}\right) J\widehat{\Delta p}_{i1}\widehat{\Delta p}_{i1}^T J^T & \dots & \sigma_{\phi_i}^2 J\widehat{\Delta p}_{i1}\widehat{\Delta p}_{iM_i}^T J^T \\ \vdots & \ddots & \vdots \\ \sigma_{\phi_i}^2 J\widehat{\Delta p}_{iM_i}\widehat{\Delta p}_{i1}^T J^T & \dots & \sigma_{\rho_i}^2 I_{2\times 2} + \left(\sigma_{\phi_i}^2 + \sigma_{\theta_i}^2 - \frac{\sigma_{\rho_i}^2}{\hat{\rho}_{iM_i}^2}\right) J\widehat{\Delta p}_{iM_i}\widehat{\Delta p}_{iM_i}^T J^T \end{bmatrix} \tag{23}$$

$$\begin{aligned} &= \sigma_{\rho_i}^2 I_{2M_i\times 2M_i} + D_i(k+1)\left(\sigma_{\theta_i}^2 I_{M_i\times M_i} + \sigma_{\phi_i}^2 \mathbf{1}_{M_i\times M_i} - \operatorname{diag}\left(\frac{\sigma_{\rho_i}^2}{\hat{\rho}_{ij}^2}\right)\right) D_i^T(k+1) \\ &= \underbrace{\sigma_{\rho_i}^2 I_{2M_i\times 2M_i} - D_i(k+1)\operatorname{diag}\left(\frac{\sigma_{\rho_i}^2}{\hat{\rho}_{ij}^2}\right) D_i^T(k+1)}_{R_1(k+1)} + \underbrace{\sigma_{\theta_i}^2 D_i(k+1) D_i^T(k+1)}_{R_2(k+1)} + \underbrace{\sigma_{\phi_i}^2 D_i(k+1)\mathbf{1}_{M_i\times M_i} D_i^T(k+1)}_{R_3(k+1)} \end{aligned} \tag{24}$$

where

$$D_i(k+1) = \begin{bmatrix} J\widehat{\Delta p}_{i1_{k+1|k}} & \dots & 0_{2\times 1} \\ \vdots & \ddots & \vdots \\ 0_{2\times 1} & \dots & J\widehat{\Delta p}_{iM_{i_{k+1|k}}} \end{bmatrix} = \mathbf{Diag}\left(J\widehat{\Delta p}_{ij_{k+1|k}}\right)$$

is a $2M_i \times M_i$ block diagonal matrix, depending on the estimated positions of the robots and landmarks. In Eq. (24) the covariance term $R_1(k+1)$ is the covariance of the error due to the noise in the range measurements, $R_2(k+1)$ is the covariance term due to the error in the bearing measurements, and $R_3(k+1)$ is the covariance term due to the error in the orientation estimates of the robot.

Since the measurements performed by different robots are independent, the covariance matrix of measurements for the entire system is given by

$$\mathbf{R}(k+1) = \mathbf{Diag}\left(\boldsymbol{R}_i(k+1)\right) = \mathbf{Diag}\left(\boldsymbol{\Xi}_{\hat{\phi}_i}^T \boldsymbol{R}_{o_i}(k+1)\boldsymbol{\Xi}_{\hat{\phi}_i}\right) = \boldsymbol{\Xi}^T(k+1)\mathbf{R}_o(k+1)\boldsymbol{\Xi}(k+1) \tag{25}$$

where $\mathbf{R}_o$ is a block diagonal matrix with block elements $\boldsymbol{R}_{o_i}$, $i = 0 \dots M$ and $\boldsymbol{R}_{o_0}$ is the covariance matrix of GPS measurement given by:

$$\boldsymbol{R}_{o_0} = \begin{bmatrix} \sigma_{GPS_x}^2 & 0 \\ 0 & \sigma_{GPS_y}^2 \end{bmatrix} \tag{26}$$

And $\boldsymbol{\Xi}_{\hat{\phi}_0}$ is the identity rotation matrix.

We now write the covariance update equation, which is

$$\mathbf{P}_{k+1|k+1} = \mathbf{P}_{k+1|k} - \mathbf{P}_{k+1|k}\mathbf{H}^T(k+1)\left(\mathbf{H}(k+1)\mathbf{P}_{k+1|k}\mathbf{H}^T(k+1) + \mathbf{R}(k+1)\right)^{-1}\mathbf{H}(k+1)\mathbf{P}_{k+1|k}$$

$$= \mathbf{P}_{k+1|k}$$
$$- \mathbf{P}_{k+1|k}\mathbf{H}_o^T\Xi(k+1)\left(\Xi^T(k+1)\mathbf{H}_o\mathbf{P}_{k+1|k}\mathbf{H}_o^T\Xi(k+1) + \Xi^T(k+1)\mathbf{R}_o(k+1)\Xi(k+1)\right)^{-1}\Xi^T(k+1)\mathbf{H}_o\mathbf{P}_{k+1|k}$$
$$= \mathbf{P}_{k+1|k} - \mathbf{P}_{k+1|k}\mathbf{H}_o^T\left(\mathbf{H}_o\mathbf{P}_{k+1|k}\mathbf{H}_o^T + \mathbf{R}_o(k+1)\right)^{-1}\mathbf{H}_o\mathbf{P}_{k+1|k} \quad (27)$$

In order to derive the last expression, property $\Xi^T(k+1) = \Xi^{-1}(k+1)$ was employed. This property is a consequence of the definition of matrix $\Xi(k+1)$, and the fact that the rotation matrices satisfy $C^T(\hat{\phi}_i) = C^{-1}(\hat{\phi}_i)$.

4 CLATT Positioning Accuracy Characterization

4.1 The Riccati Recursion

The metric we employ in order to characterize the positioning performance of the current problem is the covariance matrix of the robots and target position estimates. By combining Eqs. (8) and (27) we derive the discrete-time Riccati recursion, that describes the time evolution of the covariance matrix:

$$\mathbf{P}_{k+2|k+1} = \mathbf{P}_{k+1|k} - \mathbf{P}_{k+1|k}\mathbf{H}_o^T\left(\mathbf{H}_o\mathbf{P}_{k+1|k}\mathbf{H}_o^T + \mathbf{R}_o(k+1)\right)^{-1}\mathbf{H}_o\mathbf{P}_{k+1|k} + \mathbf{Q}(k+1)$$

This recursion provides the value of the covariance matrix at each time step, right after the propagation phase of the EKF. To simplify the notation, we set $\mathbf{P}_k = \mathbf{P}_{k+1|k}$ and $\mathbf{P}_{k+1} = \mathbf{P}_{k+2|k+1}$, and therefore we can write

$$\mathbf{P}_{k+1} = \mathbf{P}_k - \mathbf{P}_k\mathbf{H}_o^T\left(\mathbf{H}_o\mathbf{P}_k\mathbf{H}_o^T + \mathbf{R}_o(k+1)\right)^{-1}\mathbf{H}_o\mathbf{P}_k + \mathbf{Q}(k+1) \quad (28)$$

We note that the matrices $\mathbf{Q}(k+1)$ and $\mathbf{R}_o(k+1)$ in this Riccati recursion are time varying, and this does not allow the derivation of any closed form expressions for the time evolution of $\mathbf{P}_k$, in the general case. Therefore we have to derive some *bounds* for the covariance of the robots and target position estimates. The following two lemmas are the basis of our analysis:

Lemma 4.1 *If $\mathbf{R}_u$ and $\mathbf{Q}_u$ are matrices such that $\mathbf{R}_u \succeq \mathbf{R}_o(k)$ and $\mathbf{Q}_u \succeq \mathbf{Q}(k)$ for all $k \geq 0$, then the solution to the Riccati recursion*

$$\mathbf{P}^u_{k+1} = \mathbf{P}^u_k - \mathbf{P}^u_k\mathbf{H}_o^T\left(\mathbf{H}_o\mathbf{P}^u_k\mathbf{H}_o^T + \mathbf{R}_u\right)^{-1}\mathbf{H}_o\mathbf{P}^u_k + \mathbf{G}_o\mathbf{Q}_u\mathbf{G}_o^T \quad (29)$$

with the initial condition $\mathbf{P}^u_0 = \mathbf{P}_0$, satisfies $\mathbf{P}^u_k \succeq \mathbf{P}_k$ for all $k \geq 0$.

Lemma 4.2 *If $\bar{\mathbf{R}}$ and $\bar{\mathbf{Q}}$ are matrices such that $\bar{\mathbf{R}} = E\{\mathbf{R}_o(k)\}$ and $\bar{\mathbf{Q}} = E\{\mathbf{Q}(k)\}$ for all $k \geq 0$, then the solution to the Riccati recursion*

$$\bar{\mathbf{P}}_{k+1} = \bar{\mathbf{P}}_k - \bar{\mathbf{P}}_k\mathbf{H}_o^T\left(\mathbf{H}_o\bar{\mathbf{P}}_k\mathbf{H}_o^T + \bar{\mathbf{R}}\right)^{-1}\mathbf{H}_o\bar{\mathbf{P}}_k + \bar{\mathbf{Q}} \quad (30)$$

with the initial condition $\bar{\mathbf{P}}_0 = \mathbf{P}_0$, satisfies $\bar{\mathbf{P}}_k \succeq E\{\mathbf{P}_k\}$ for all $k \geq 0$.

Essentially, Lemma 4.1 maintains that in order to derive an upper bound on the *worst-case* covariance matrix of the position estimates of robots and target, it suffices to derive *upper bounds* for the covariance matrices of the system and measurement noise, and to solve a *constant coefficient* Riccati recursion. Similarly, Lemma 4.2 states that an upper bound on the *expected* positioning uncertainty of the robots and the target is determined as the solution of a constant coefficient Riccati recursion, where the covariance matrices of the system and measurement noise have been replaced by their *average* values. The proofs for these lemmas are given in Appendices A and B respectively. In the remainder of this section, we derive appropriate upper bounds, as well as the average values of the matrices $\mathbf{Q}(k)$ and $\mathbf{R}_o(k)$ respectively.

- **Derivation of upper bounds for $\mathbf{Q}(k)$ and $\mathbf{R}_o(k)$**

In order to derive an upper bound for the covariance matrix $\mathbf{Q}_r(k)$ we note that (cf. Eq. (7))

$$\mathbf{Q}(k) = \begin{bmatrix} \mathbf{Diag}(Q_{r_i}(k)) & \mathbf{0}_{2M\times 2N} \\ \mathbf{0}_{2N\times 2M} & I_N \otimes Q_T \end{bmatrix} \quad (31)$$

where for $i = 1 \dots M$:

$$Q_{r_i}(k) = C(\hat{\phi}_i(k)) \begin{bmatrix} \delta t^2 \sigma_{V_i}^2 & 0 \\ 0 & \delta t^2 V_{m_i}^2(k) \sigma_{\phi_i}^2 \end{bmatrix} C^T(\hat{\phi}_i(k))$$

and,

$$Q_T = \begin{bmatrix} \sigma_{T_x}^2 & 0 \\ 0 & \sigma_{T_y}^2 \end{bmatrix}$$

From the properties of rotation matrices it is known that $C^{-1}(\hat{\phi}_i(k)) = C^T(\hat{\phi}_i(k))$, and thus $Q_{r_i}(k)$ is related by a similarity transformation to the matrix

$$\begin{bmatrix} \delta t^2 \sigma_{V_i}^2 & 0 \\ 0 & \delta t^2 V_{m_i}^2(k) \sigma_{\phi_i}^2 \end{bmatrix}$$

for $i = 1 \dots M$. So the eigenvalues of $Q_{r_i}(k), i = 1 \dots M$ are $\delta t^2 \sigma_{V_i}^2$ and $\delta t^2 V_{m_i}^2(k) \sigma_{\phi_i}^2$. We assume that the velocity of each robot is approximately constant, and equal to V_i, and denote

$$q_i = \max\left(\delta t^2 \sigma_{V_i}^2, \delta t^2 V_{m_i}^2(k) \sigma_{\phi_i}^2\right) \simeq \max\left(\delta t^2 \sigma_{V_i}^2, \delta t^2 V_i^2 \sigma_{\phi_i}^2\right)\ ,\ i = 1 \dots M \tag{32}$$

This definition states that q_i is the largest eigenvalue of $Q_{r_i}(k)$, and therefore

$$Q_{r_i}(k) \preceq C(\hat{\phi}_i(k)) q_i I_{2\times 2} C^T(\hat{\phi}_i(k)) = q_i I_{2\times 2} = Q_{u_i} \tag{33}$$

for $i = 1 \dots M$. Considering that Q_T is fixed and independent from the state vector variables, it can be used directly as the upper bound:

$$Q_{u_T} = Q_T = \begin{bmatrix} \sigma_{T_x}^2 & 0 \\ 0 & \sigma_{T_y}^2 \end{bmatrix} \tag{34}$$

Finally the upper bound for the covariance of the entire system can be written as:

$$\mathbf{Q}_u = \begin{bmatrix} \mathbf{Diag}(Q_{u_i}(k)) & \mathbf{0}_{2M\times 2N} \\ \mathbf{0}_{2N\times 2M} & I_N \otimes Q_{u_T} \end{bmatrix} \tag{35}$$

The upper bound on $\mathbf{R}_o(k)$ is obtained by considering each if its block diagonal elements, $\boldsymbol{R}_{o_i}(k)$. Referring to Eq. (24), we examine the terms $R_1(k)$, $R_2(k)$ and $R_3(k)$ separately for range measurement (GPS measurement will be considered later). The term expressing the effect of the noise in the range measurements is

$$R_1(k) = \sigma_{\rho_i}^2 I_{2M_i\times 2M_i} - D_i(k) \operatorname{diag}\left(\frac{\sigma_{\rho_i}^2}{\hat{\rho}_{ij}^2}\right) D_i^T(k) \preceq \sigma_{\rho_i}^2 I_{2M_i\times 2M_i} \tag{36}$$

The last matrix inequality follows from the fact that the term being subtracted from $\sigma_{\rho_i}^2 I_{2M_i\times 2M_i}$ is a positive semidefinite matrix. The covariance term due to the noise in the bearing measurement is

$$\begin{aligned} R_2(k) &= \sigma_{\theta_i}^2 D_i(k) D_i^T(k) \\ &= \sigma_{\theta_i}^2 \mathbf{Diag}\left(\hat{\rho}_{ij}^2 \begin{bmatrix} \sin^2(\hat{\theta}_{ij}) & \sin(\hat{\theta}_{ij})\cos(\hat{\theta}_{ij}) \\ \sin(\hat{\theta}_{ij})\cos(\hat{\theta}_{ij}) & \cos^2(\hat{\theta}_{ij}) \end{bmatrix}\right) \\ &\preceq \sigma_{\theta_i}^2 \mathbf{Diag}\left(\hat{\rho}_{ij}^2 I_{2\times 2}\right) \\ &\preceq \sigma_{\theta_i}^2 \rho_o^2 I_{2M_i\times 2M_i} \end{aligned} \tag{37}$$

where ρ_o is the maximum range at which a measurement can occur, determined either by the characteristics of the robots' sensors or by the properties of the area in which the robots move. The other covariance term is due to the error in the orientation of the measuring robot, $R_3(k) = \sigma_{\phi_i}^2 D_i(k) \mathbf{1}_{M_i\times M_i} D_i^T(k)$. Calculation of the eigenvalues of the matrices $\mathbf{1}_{M_i\times M_i}$ and $I_{M_i\times M_i}$ verifies that $\mathbf{1}_{M_i\times M_i} \preceq M_i I_{M_i\times M_i}$, and thus we can write $R_3(k) \preceq M_i \sigma_{\phi_i}^2 D_i(k) D_i^T(k)$. By derivations analogous to those employed to yield an upper bound for $R_2(k)$, we can show that

$$R_3(k) \preceq M_i \sigma_{\phi_i}^2 \rho_o^2 I_{2M_i\times 2M_i} \tag{38}$$

By combining this result with those of Eqs. (36), (37), (38) we can write $\boldsymbol{R}_{o_i}(k) = R_1(k) + R_2(k) + R_3(k) \preceq \mathbf{R}_i^u$, $i = 1 \ldots M$, where

$$\boldsymbol{R}_i^u = \left(\sigma_{\rho_i}^2 + M_i \sigma_{\phi_i}^2 \rho_o^2 + \sigma_{\theta_i}^2 \rho_o^2\right) I_{2M_i \times 2M_i} = r_i I_{2M_i \times 2M_i} \tag{39}$$

with

$$r_i = \sigma_{\rho_i}^2 + (M_i \sigma_{\phi_i}^2 + \sigma_{\theta_i}^2)\rho_0^2 \tag{40}$$

For the R_{o_0}, the GPS measurement covariance is constant and equal to its upper bound:

$$\boldsymbol{R}_0^u = \begin{bmatrix} \sigma_{GPS_x}^2 & 0 \\ 0 & \sigma_{GPS_y}^2 \end{bmatrix}$$

Thus an upper bound for $\mathbf{R}_o$ is given by

$$\mathbf{R}_o(k) = \mathbf{Diag}(\boldsymbol{R}_{o_i}(k)) \preceq \mathbf{Diag}(\boldsymbol{R}_i^u) = \mathbf{R}_u \tag{41}$$

- **Derivation of the Expected Values of $\mathbf{Q}_r(k)$ and $\mathbf{R}_o(k)$**

  In order to derive the average value of $\mathbf{Q}_r(k)$ we note that for $i = 1 \ldots M$

$$\begin{aligned} Q_{r_i}(k) &= C(\hat{\phi}_i(k)) \begin{bmatrix} \delta t^2 \sigma_{V_i}^2 & 0 \\ 0 & \delta t^2 V_{m_i}^2(k) \sigma_{\phi_i}^2 \end{bmatrix} C^T(\hat{\phi}_i(k)) \\ &= \delta t^2 \begin{bmatrix} \sigma_{V_i}^2 \cos^2(\hat{\phi}_i) + V_{m_i}^2(k)\sigma_{\phi_i}^2 \sin^2(\hat{\phi}_i) & \left(\sigma_{V_i}^2 - V_{m_i}^2(k)\sigma_{\phi_i}^2\right) \sin(\hat{\phi}_i)\cos(\hat{\phi}_i) \\ \left(\sigma_{V_i}^2 - V_{m_i}^2(k)\sigma_{\phi_i}^2\right) \sin(\hat{\phi}_i)\cos(\hat{\phi}_i) & \sigma_{V_i}^2 \sin^2(\hat{\phi}_i) + V_{m_i}^2(k)\sigma_{\phi_i}^2 \cos^2(\hat{\phi}_i) \end{bmatrix} \end{aligned}$$

and therefore, by averaging over all values of orientation, the expected value of $Q_{r_i}(k)$ is derived:

$$E\{Q_{r_i}(k)\} = \delta t^2 \frac{\sigma_V^2 + V_i^2 \sigma_{\phi_i}^2}{2} I_{2\times 2} = \bar{q}_i I_{2\times 2}$$

The system noise covariance of the target is assumed to be constant, so it is the same as its expected value

$$E\{Q_T\} = \begin{bmatrix} \sigma_{T_x}^2 & 0 \\ 0 & \sigma_{T_y}^2 \end{bmatrix}$$

Thus,

$$\bar{\mathbf{Q}} = E\{\mathbf{Q}(k)\} = \begin{bmatrix} \mathbf{Diag}\left(E\{Q_{r_i}(k)\}\right) & \mathbf{0}_{3M\times 3N} \\ \mathbf{0}_{3N\times 3M} & I_N \otimes E\{Q_T\} \end{bmatrix} \tag{42}$$

$$\tag{43}$$

Now the average values of the matrices $\boldsymbol{R}_{o_i}(k)$, $i = 1 \ldots M$ need to be determined, in order to compute $E\{\mathbf{R}_o'(k)\}$ (GPS measurement will be considered consequently). From Eq. (24) we note that evaluation of the average value of $\boldsymbol{R}_{o_i}(k)$ requires the computation of the expected values of the following terms:

$$T_1 = \frac{\widehat{\Delta p}_{ij} \widehat{\Delta p}_{ij}^T}{\hat{\rho}_{ij}^2}, \quad T_2 = \widehat{\Delta p}_{ij} \widehat{\Delta p}_{ij}^T, \quad \text{and} \quad T_3 = \widehat{\Delta p}_{ij} \widehat{\Delta p}_{iT}^T \tag{44}$$

for $j, \ell = 1 \ldots M_i$. The average value of T_1 is easily derived by employing the polar coordinate description of the vector $\widehat{\Delta p}_{ij}$ in terms of $\hat{\rho}_{ij}$ and $\hat{\varphi}_{ij} = \hat{\phi}_i(k+1) + \hat{\theta}_{ij}$, which yields (cf. Eq. (19))

$$\begin{aligned} T_1 &= \frac{\widehat{\Delta p}_{ij} \widehat{\Delta p}_{ij}^T}{\hat{\rho}_{ij}^2} \\ &= \frac{1}{\hat{\rho}_{ij}^2} \begin{bmatrix} \hat{\rho}_{ij}^2 \cos^2(\hat{\varphi}_{ij}) & \hat{\rho}_{ij}^2 \sin(\hat{\varphi}_{ij})\cos(\hat{\varphi}_{ij}) \\ \hat{\rho}_{ij}^2 \sin(\hat{\varphi}_{ij})\cos(\hat{\varphi}_{ij}) & \hat{\rho}_{ij}^2 \sin^2(\hat{\varphi}_{ij}) \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} \hat{\rho}_{ij}^2 \cos^2(\hat{\varphi}_{ij}) & \hat{\rho}_{ij}^2 \sin(\hat{\varphi}_{ij}) \cos(\hat{\varphi}_{ij}) \\ \hat{\rho}_{ij}^2 \sin(\hat{\varphi}_{ij}) \cos(\hat{\varphi}_{ij}) & \hat{\rho}_{ij}^2 \sin^2(\hat{\varphi}_{ij}) \end{bmatrix} \tag{45}$$

From the last expression we conclude that for any probability density function that guarantees a uniform distribution for the bearing angle on $[0\ ,\ 2\pi]$, the average value of the term T_1 is

$$E\{T_1\} \;=\; \frac{1}{2} I_{2\times 2}$$

In order to compute the expected value of the terms T_2 and T_3, we assume that the robots and the target are located in a square arena of side α, and that their positions are described by uniformly distributed random variables in the interval $[-\alpha/2, \alpha/2]$. Thus we can write,

$$E\{T_2\} = E\left\{ \begin{bmatrix} \widehat{\Delta x}_{ij}^2 & \widehat{\Delta x}_{ij} \widehat{\Delta y}_{ij} \\ \widehat{\Delta y}_{ij} \widehat{\Delta x}_{ij} & \widehat{\Delta y}_{ij}^2 \end{bmatrix} \right\} \tag{46}$$

The non-diagonal elements of the above matrix are always zero as we suppose the variables are independent. For example:

$$E\{\widehat{\Delta x}_{ij} \widehat{\Delta y}_{ij}\} = E\{\widehat{\Delta x}_{ij}\} E\{\widehat{\Delta y}_{ij}\} = 0$$

The diagonal elements can be computed readily:

$$E\{\widehat{\Delta x}_{ij}^2\} = E\{x_i^2 - 2x_i x_j + x_j^2\} = \frac{\alpha^2}{6}$$

Therefore,

$$E\{T_2\} \;=\; \frac{\alpha^2}{6} I_{2\times 2}$$

Similarly,

$$\begin{aligned} E\{T_3\} = E\{\widehat{\Delta p}_{ij} \widehat{\Delta p}_{iT}^T\} &= E\left\{ \begin{bmatrix} \widehat{\Delta x}_{ij} \widehat{\Delta x}_{iT} & \widehat{\Delta x}_{ij} \widehat{\Delta y}_{iT} \\ \widehat{\Delta y}_{ij} \widehat{\Delta x}_{iT} & \widehat{\Delta y}_{ij} \widehat{\Delta y}_{iT} \end{bmatrix} \right\} \\ &= \begin{bmatrix} E\{x_j x_T - x_i x_T - x_j x_i + x_i^2\} & E\{x_j y_T - x_j y_i - x_i y_T + x_i y_i\} \\ E\{y_j x_T - y_j x_i - y_i x_T + y_i x_i\} & E\{y_j y_T - y_i y_T - y_j y_i + y_i^2\} \end{bmatrix} \\ &= \begin{bmatrix} E\{x_i^2\} & 0 \\ 0 & E\{y_i^2\} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\alpha^2}{12} & 0 \\ 0 & \frac{\alpha^2}{12} \end{bmatrix} \\ &= \frac{\alpha^2}{12} I_{2\times 2} \end{aligned}$$

These results enable us to obtain the average value of the matrices $\boldsymbol{R}_{o_i}(k)$, $i = 1 \ldots M$. Employing the linearity of the expectation operator yields

$$\begin{aligned} \bar{\mathbf{R}}_i &= E\{\boldsymbol{R}_{o_i}(k)\} \\ &= \begin{bmatrix} \left(\frac{1}{2}\sigma_{\rho_i}^2 + \frac{\alpha^2}{6}\sigma_{\phi_i}^2 + \frac{\alpha^2}{6}\sigma_{\theta_i}^2\right) I_{2\times 2} & \cdots & \frac{\alpha^2}{12}\sigma_{\phi_i}^2 I_{2\times 2} \\ \vdots & \ddots & \vdots \\ \frac{\alpha^2}{12}\sigma_{\phi_i}^2 I_{2\times 2} & \cdots & \left(\frac{1}{2}\sigma_{\rho_i}^2 + \frac{\alpha^2}{6}\sigma_{\phi_i}^2 + \frac{\alpha^2}{6}\sigma_{\theta_i}^2\right) I_{2\times 2} \end{bmatrix} \\ &= \left(\frac{1}{2}\sigma_{\rho_i}^2 + \frac{\alpha^2}{12}\sigma_{\phi_i}^2 + \frac{\alpha^2}{6}\sigma_{\theta_i}^2\right) I_{2M_i\times 2M_i} + \frac{\alpha^2}{12}\sigma_{\phi_i}^2 \left(\mathbf{1}_{M_i\times M_i} \otimes I_{2\times 2}\right) \end{aligned}$$

And the covariance of GPS measurement noise is constant, so its expected value is:

$$\bar{\mathbf{R}}_0 = E\{\boldsymbol{R}_{o_0}(k)\} = \begin{bmatrix} \sigma^2_{GPS_x} & 0 \\ 0 & \sigma^2_{GPS_y} \end{bmatrix} \tag{47}$$

The average value of $\mathbf{R}_o(k)$ is therefore

$$\begin{aligned} \bar{\mathbf{R}} &= E\{\mathbf{R}_o(k)\} \\ &= \mathbf{Diag}(\bar{\mathbf{R}}_i) \end{aligned} \tag{48}$$

4.2 Evaluation of the Upper Bounds with Finite Target Noise Covariance

Lemmas 4.1 and 4.2 allow the evaluation of upper bounds on the worst case uncertainty and on the average uncertainty of the position estimates of the robots and the target , at *any* time instant after the deployment of the robot team. This can be achieved, for example, by numerical evaluation of the solution to the recursions in Eqs. (29) and (30) respectively. For many applications, it is of interest however, to study the steady-state behavior of the positioning uncertainty in of CLATT. For this reason, we now derive the steady-state values of the solutions to the recursions (29) and (30). By "steady-state values" we refer to the values of the covariance matrix after a sufficient time has elapsed, enough for the the initial transient phenomena in the solutions to subside. The steady state solutions are derived by evaluating the limit of $\mathbf{P}^u_k$ and $\bar{\mathbf{P}}_k$ as $k \to \infty$. We note at this point that the Riccati recursions of Eqs. (29) and (30) essentially describe the time evolution of the covariance of the position estimates in two hypothetical CLATT scenarios, where the system model is a Linear Time Invariant (LTI) one. Therefore, the problem of computing the upper bounds on the steady state positioning uncertainty in CLATT reduces to the problem of *determining the steady state covariance matrix for a LTI CLATT system model*.

To avoid redundant derivations, in the following we will solve for the steady state solution of the following Riccati recursion:

$$\mathbf{P}^s_{k+1} = \mathbf{P}^s_k - \mathbf{P}^s_k\mathbf{H}'^T_o \left(\mathbf{H}'_o\mathbf{P}^s_k\mathbf{H}'^T_o + \mathbf{R}_s\right)^{-1} \mathbf{H}'_o\mathbf{P}^s_k + \mathbf{Q}_s \tag{49}$$

After deriving the steady state solution of this recursion, we employ the substitutions

$$\mathbf{R}_s \to \mathbf{R}_u, \qquad \mathbf{Q}_s \to \mathbf{Q}_u$$

and

$$\mathbf{R}_s \to \bar{\mathbf{R}}\ , \qquad \mathbf{Q}_s \to \bar{\mathbf{Q}}$$

in order to obtain the steady state solutions of the Riccati recursions of Lemmas (4.1) and (4.2) respectively.

We first note that the Riccati recursion in Eq. (49) can be reformulated as follows, by use of the matrix inversion lemma (cf. Appendix C):

$$\begin{aligned} \mathbf{P}^s_{k+1} &= \mathbf{P}^s_k - \mathbf{P}^s_k\mathbf{H}^T_o \left(\mathbf{H}_o\mathbf{P}^s_k\mathbf{H}^T_o + \mathbf{R}_s\right)^{-1} \mathbf{H}_o\mathbf{P}^s_k + \mathbf{Q}_s \\ &= \mathbf{P}^s_k \left(I_{2N\times 2N} + \mathbf{H}^T_o\mathbf{R}^{-1}_s\mathbf{H}_o\mathbf{P}^s_k\right)^{-1} + \mathbf{Q}_s \end{aligned} \tag{50}$$

The derivations are simplified by defining the *normalized* covariance matrix as

$$\mathbf{P}_{n_k} = \mathbf{Q}^{-1/2}_s\mathbf{P}^s_k\mathbf{Q}^{-1/2}_s \tag{51}$$

Pre- and post-multiplying Eq. (50) by $\mathbf{Q}^{-1/2}_s$, and simple algebraic manipulation yields

$$\mathbf{P}_{n_{k+1}} = \mathbf{P}_{n_k}\left(I_{2N\times 2N} + \mathbf{C}_s\mathbf{P}_{n_k}\right)^{-1} + I_{2N\times 2N} \tag{52}$$

where

$$\mathbf{C}_s = \mathbf{Q}^{1/2}_s\mathbf{H}^T_o\mathbf{R}^{-1}_s\mathbf{H}_o\mathbf{Q}^{1/2}_s$$

Note that the only parameter in the Riccati recursion (52) is the matrix $\mathbf{C}_s$, which contains the main parameters that characterize the localization performance of the robotic team. The eigenvalues of this matrix, which are studied in Appendix D, are in close relation with the type and number of exteroceptive measurements recorded by the robots of the team, and determine the properties of the upper bound on the steady-state positioning uncertainty. To further simplify the derivations, we denote the Singular Value Decomposition (SVD) of $\mathbf{C}_s$ as

$$\mathbf{C}_s = \mathbf{U}_s \,\mathrm{diag}(\lambda_i)\mathbf{U}_s^T = \mathbf{U}_s\Lambda\mathbf{U}_s^T$$

and substituting in Eq. (52) we obtain[2]

$$\begin{aligned}
\mathbf{P}_{n_{k+1}} &= \mathbf{P}_{n_k}\left(I + \mathbf{U}_s\Lambda\mathbf{U}_s^T\mathbf{P}_{n_k}\right)^{-1} + I \Rightarrow \\
\mathbf{U}_s^T\mathbf{P}_{n_{k+1}}\mathbf{U}_s &= \mathbf{U}_s^T\mathbf{P}_{n_k}\mathbf{U}_s\mathbf{U}_s^T\left(I + \mathbf{U}_s\Lambda\mathbf{U}_s^T\mathbf{P}_{n_k}\right)^{-1}\mathbf{U}_s + I \Rightarrow \\
\mathbf{U}_s^T\mathbf{P}_{n_{k+1}}\mathbf{U}_s &= \mathbf{U}_s^T\mathbf{P}_{n_k}\mathbf{U}_s\left(I + \Lambda\mathbf{U}_s^T\mathbf{P}_{n_k}\mathbf{U}_s\right)^{-1} + I
\end{aligned}$$

We define

$$\mathbf{P}_{nn_k} = \mathbf{U}_s^T\mathbf{P}_{n_k}\mathbf{U}_s \tag{53}$$

and we obtain the recursion

$$\mathbf{P}_{nn_{k+1}} = \mathbf{P}_{nn_{k+1}}\left(I + \Lambda\mathbf{P}_{nn_{k+1}}\right)^{-1} + I \tag{54}$$

This form of the recursion is simpler, since now the only parameter is the diagonal matrix of the eigenvalues of $\mathbf{C}_s$.

4.2.1 Observable system

As one of the robots is equipped with GPS, the system is observable [16], and therefore the covariance of the robots and target position estimation remains bounded at steady state. For this case, it is shown in Appendix D that $\mathrm{rank}(\mathbf{C}_s) = 2M+2$, and therefore all the singular values of $\mathbf{C}_s$ are positive.

Since we are dealing with an observable system, the solution to Eq. (54) will converge to a constant value at steady state, determined by solving the Discrete Algebraic Riccati Equation (DARE):

$$\mathbf{P}_{nn_{ss}} = \mathbf{P}_{nn_{ss}}\left(I + \Lambda\mathbf{P}_{nn_{ss}}\right)^{-1} + I$$

Since the system is both controllable and observable, the solution of the above DARE is unique [17]. Therefore, we can "guess" a solution, and if it satisfies the DARE, we can be assured that this is the only possible solution. We now assume a diagonal form for $\mathbf{P}_{nn_{ss}}$. In that case, all the matrices in the above DARE are diagonal, and thus we obtain the following set of $2M+2$ independent equations:

$$P_{nn_{ss}}(i,i) = \frac{P_{nn_{ss}}(i,i)}{1+\lambda_i P_{nn_{ss}}(i,i)} + 1, \quad i = 1\dots 2M+2 \tag{55}$$

Whose solution is given by

$$P_{nn_{ss}}(i,i) = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{\lambda_i}}$$

By substitution of this result in Eqs. (53) and (51), we obtain the steady state solution to the Riccati recursion (49):

$$\mathbf{P}_{ss}^s = \mathbf{Q}_s^{1/2}\mathbf{U}_s\,\mathrm{diag}\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{\lambda_i}}\right)\mathbf{U}_s^T\mathbf{Q}_s^{1/2} \tag{56}$$

Finally, from this result, by setting

$$\mathbf{R}_s \to \mathbf{R}_u, \qquad \mathbf{Q}_s \to \mathbf{Q}_u$$

and

$$\mathbf{R}_s \to \bar{\mathbf{R}}\,, \qquad \mathbf{Q}_s \to \bar{\mathbf{Q}}$$

we can derive the following lemmas:

[2]To make the notation less cumbersome, we hereafter omit the dimension index from the identity matrices, whenever their dimension is equal to the dimension of the state covariance matrix. I.e., from this point on, $I = I_{2N\times 2N}$.

Lemma 4.3 *The steady state covariance of the position estimates for a team of robots performing CLATT, when at least one robot has access to absolute positioning information is bounded above by the matrix*

$$\mathbf{P}^u_{ss} \quad = \quad \mathbf{Q}_u^{1/2}\mathbf{U}_u \operatorname{diag}\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{\lambda_{u_i}}}\right) \mathbf{U}_u^T \mathbf{Q}_u^{1/2} \tag{57}$$

where we have denoted the singular value decomposition of $\mathbf{C}_u = \mathbf{Q}_u^{1/2}\mathbf{H}_o^T\mathbf{R}_u^{-1}\mathbf{H}_o\mathbf{Q}_u^{1/2}$ *as* $\mathbf{C}_u = \mathbf{U}_u \operatorname{diag}(\lambda_{u_i})\mathbf{U}_u^T$.

Lemma 4.4 *The expected steady state covariance of the position estimates for a team of robots performing CL, when at least one robot has access to absolute positioning information is bounded above by the matrix*

$$\bar{\mathbf{P}}_{ss} \quad = \quad \bar{\mathbf{Q}}^{1/2}\bar{\mathbf{U}} \operatorname{diag}\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{\bar{\lambda}_i}}\right) \bar{\mathbf{U}}^T \bar{\mathbf{Q}}^{1/2} \tag{58}$$

where we have denoted the singular value decomposition of $\bar{\mathbf{C}} = \bar{\mathbf{Q}}^{1/2}\mathbf{H}_o^T\bar{\mathbf{R}}^{-1}\mathbf{H}_o\bar{\mathbf{Q}}^{1/2}$ *as* $\bar{\mathbf{C}} = \bar{\mathbf{U}} \operatorname{diag}(\bar{\lambda}_i)\bar{\mathbf{U}}^T$.

At this point we should note that the upper bounds on the steady-state uncertainty depend on the topology of the RPMG and the accuracy of the proprioceptive and exteroceptive sensors of the robots. However, the steady-state uncertainty is independent of the initial covariance of the robots, which comes as no surprise, since the system is observable.

4.3 Evaluation of the Upper Bounds with Infinite Target Noise Covariance

The zero velocity target model is the simplest target tracking model. Instead of using more complex target models, the target noise can be increased so that higher order models are included in this simple model. In this section, we consider CLATT when target noise approach infinity and we will find a performance bound for the robots and target positions.

Starting from equation (50), the Riccati equation can be arranged as:

$$\begin{aligned}
\mathbf{P}_{k+1} &= \mathbf{P}_k - \mathbf{P}_k\mathbf{H}_o^T\left(\mathbf{H}_o\mathbf{P}_k\mathbf{H}_o^T + \mathbf{R}_s\right)^{-1}\mathbf{H}_o\mathbf{P}_k + \mathbf{Q}_s \\
&= \mathbf{P}_k\left(I_{2N\times 2N} + \mathbf{H}_o^T\mathbf{R}_s^{-1}\mathbf{H}_o\mathbf{P}_k\right)^{-1} + \mathbf{Q}_s \\
&= \left(\mathbf{P}_k^{-1} + \mathbf{H}_o^T\mathbf{R}_s^{-1}\mathbf{H}_o\right)^{-1} + \mathbf{Q}_s
\end{aligned} \tag{59}$$

where the superscript s on $\mathbf{P}$ is ignored for simplicity. The term $\mathbf{\Psi} = \mathbf{H}_o^T\mathbf{R}_s^{-1}\mathbf{H}_o$ is the new information added after each measurement step and is constant as long as RPMG is not changed.

Considering

$$\mathbf{Q}_s = \begin{bmatrix}\mathbf{Q}_{rr} & 0 \\ 0 & \mathbf{Q}_{tt}\end{bmatrix} \quad , \quad Q_{tt} = \mu I_{2\times 2} \tag{60}$$

where $\mu \to \infty$, $\mathbf{P}_{k+1}$ is:

$$\begin{aligned}
\begin{bmatrix}\mathbf{P}_{rr} & \mathbf{P}_{rt} \\ \mathbf{P}_{tr} & \mathbf{P}_{tt}\end{bmatrix}_{(k+1)} &= \lim_{\mu\to\infty}\left(\begin{bmatrix}\mathbf{P}_{rr} & \mathbf{P}_{rt} \\ \mathbf{P}_{tr} & \mathbf{P}_{tt} + \mu I_{2\times 2}\end{bmatrix}_{(k)}^{-1} + \begin{bmatrix}\mathbf{\Psi}_{11} & \mathbf{\Psi}_{12} \\ \mathbf{\Psi}_{21} & \mathbf{\Psi}_{22}\end{bmatrix}\right)^{-1} + \mathbf{Q}_s \\
&= \left(\begin{bmatrix}\mathbf{P}_{rr_k}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix} + \begin{bmatrix}\mathbf{\Psi}_{11} & \mathbf{\Psi}_{12} \\ \mathbf{\Psi}_{21} & \mathbf{\Psi}_{22}\end{bmatrix}\right)^{-1} + \mathbf{Q}_s \\
&= \begin{bmatrix}\mathbf{P}_{rr_k}^{-1} + \mathbf{\Psi}_{11} & \mathbf{\Psi}_{12} \\ \mathbf{\Psi}_{21} & \mathbf{\Psi}_{22}\end{bmatrix}^{-1} + \mathbf{Q}_s \\
&= \begin{bmatrix}\left(\mathbf{P}_{rr_k}^{-1} + \mathbf{\Psi}_{11} - \mathbf{\Psi}_{12}\mathbf{\Psi}_{22}^{-1}\mathbf{\Psi}_{21}\right)^{-1} & \mathbf{\Upsilon} \\ \mathbf{\Upsilon}^T & \left(\mathbf{\Psi}_{22} - \mathbf{\Psi}_{21}\left((\mathbf{P}_{rr_k})^{-1} + \mathbf{\Psi}_{11}\right)^{-1}\mathbf{\Psi}_{12}\right)^{-1}\end{bmatrix} + \mathbf{Q}_s
\end{aligned} \tag{61}$$

So the Riccati equation for the target and the robots can be separated. Given that matrix $\mathbf{\Psi}$ is constant, performance of the target position estimation right after update is constant:

$$\mathbf{P}_{tt} = \left(\mathbf{\Psi}_{22} - \mathbf{\Psi}_{21}\left((\mathbf{P}_{rr_k})^{-1} + \mathbf{\Psi}_{11}\right)^{-1}\mathbf{\Psi}_{12}\right)^{-1} \tag{62}$$

Clearly this formula is not applicable after propagation and position covariance of the target will be infinite at that time. The recursive formula for the performance bound of position estimation of the robots is:

$$\begin{aligned}\mathbf{P}_{rr_{k+1}} &= \left(\mathbf{P}_{rr_k}^{-1} + \mathbf{\Psi}_{rr}\right)^{-1} + \mathbf{Q}_{rr} \\ &= \mathbf{P}_{rr_k}\left(I_{2N\times 2N} + \mathbf{\Psi}_{rr}\mathbf{P}_{rr_k}\right)^{-1} + \mathbf{Q}_{rr}\end{aligned} \tag{63}$$

where

$$\mathbf{\Psi}_{rr} = \mathbf{\Psi}_{11} - \mathbf{\Psi}_{12}\mathbf{\Psi}_{22}^{-1}\mathbf{\Psi}_{21} \tag{64}$$

Eq. (63) is similar to Eq. (50) and it can be solved in the same way:

$$\mathbf{P}_{n_k} = \mathbf{Q}_{rr}^{-1/2}\mathbf{P}_{rr_k}\mathbf{Q}_{rr}^{-1/2} \tag{65}$$

Pre- and post-multiplying Eq. (63) by $\mathbf{Q}_s^{-1/2}$, and simple algebraic manipulation yields

$$\mathbf{P}_{n_{k+1}} = \mathbf{P}_{n_k}\left(I_{2N\times 2N} + \mathbf{C}_{rr}\mathbf{P}_{n_k}\right)^{-1} + I_{2N\times 2N} \tag{66}$$

where

$$\mathbf{C}_{rr} = \mathbf{Q}_{rr}^{1/2}\mathbf{\Psi}_{rr}\mathbf{Q}_{rr}^{1/2}$$

$$\mathbf{C}_{rr} = \mathbf{U}_{rr}\,\mathrm{diag}(\lambda_i)\mathbf{U}_{rr}^T = \mathbf{U}_{rr}\Lambda\mathbf{U}_{rr}^T \tag{67}$$

At this point it should be noted that all the eigenvalues of $\mathbf{\Psi}_{rr}$ are positive because $\mathbf{H}_o^T\mathbf{R}_s^{-1}\mathbf{H}_o$ is positive definite as studied in Appendix D and by using the rule:

$$\det\mathbf{A} = \det\mathbf{A}_{11}\det\left(\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\right)$$

where

$$\mathbf{A} = \begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22}\end{bmatrix}$$

Now, using similar mathematical manipulation to Eqs. (50)-(56), the solution to this problem can be found as:

$$\mathbf{P}_{rr} = \mathbf{Q}_r^{1/2}\mathbf{U}_r\,\mathrm{diag}\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{\lambda_i}}\right)\mathbf{U}_r^T\mathbf{Q}_r^{1/2} \tag{68}$$

A Upper Bound Riccati Recursion

In this appendix we prove that if $\mathbf{R}'_u \succeq \mathbf{R}'_o(k)$ and $\mathbf{Q}_u \succeq \mathbf{Q}_r(k)$ for all $k \geq 0$, then the solutions to the following two Riccati recursions

$$\mathbf{P}_{k+1} = \mathbf{P}_k - \mathbf{P}_k \mathbf{H}_o'^T \left(\mathbf{H}_o' \mathbf{P}_k \mathbf{H}_o'^T + \mathbf{R}'_o(k+1) \right)^{-1} \mathbf{H}'_o \mathbf{P}_k + \mathbf{G}_o \mathbf{Q}_r(k+1) \mathbf{G}_o^T \tag{69}$$

and

$$\mathbf{P}^u_{k+1} = \mathbf{P}^u_k - \mathbf{P}^u_k \mathbf{H}_o'^T \left(\mathbf{H}'_o \mathbf{P}^u_k \mathbf{H}_o'^T + \mathbf{R}'_u \right)^{-1} \mathbf{H}'_o \mathbf{P}^u_k + \mathbf{G}_o \mathbf{Q}_u \mathbf{G}_o^T \tag{70}$$

with the *same* initial condition, $\mathbf{P}_0$, satisfy $\mathbf{P}^u_k \succeq \mathbf{P}_k$ for all $k \geq 0$. The proof is carried out by induction, and requires the following two intermediate results:

- **Monotonicity with respect to the measurement covariance matrix**

If $\mathbf{R}_1 \succeq \mathbf{R}_2$, then for any $\mathbf{P} \succeq 0$

$$\mathbf{P} - \mathbf{P}\mathbf{H}^T \left(\mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R}_1 \right)^{-1} \mathbf{H}\mathbf{P} + \mathbf{Q}_o \succeq \mathbf{P} - \mathbf{P}\mathbf{H}^T \left(\mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R}_2 \right)^{-1} \mathbf{H}\mathbf{P} + \mathbf{Q}_o \tag{71}$$

This statement is proven by taking into account the properties of linear matrix inequalities:

$$\begin{aligned}
\mathbf{R}_1 &\succeq \mathbf{R}_2 \Rightarrow \\
\mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R}_1 &\succeq \mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R}_2 \Rightarrow \\
\left(\mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R}_1 \right)^{-1} &\preceq \left(\mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R}_2 \right)^{-1} \Rightarrow \\
\mathbf{P}\mathbf{H}^T \left(\mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R}_1 \right)^{-1} \mathbf{H}\mathbf{P} &\preceq \mathbf{P}\mathbf{H}^T \left(\mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R}_2 \right)^{-1} \mathbf{H}\mathbf{P} \Rightarrow \\
-\mathbf{P}\mathbf{H}^T \left(\mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R}_1 \right)^{-1} \mathbf{H}\mathbf{P} &\succeq -\mathbf{P}\mathbf{H}^T \left(\mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R}_2 \right)^{-1} \mathbf{H}\mathbf{P} \Rightarrow \\
\mathbf{P} - \mathbf{P}\mathbf{H}^T \left(\mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R}_1 \right)^{-1} \mathbf{H}\mathbf{P} + \mathbf{Q}_o &\succeq \mathbf{P} - \mathbf{P}\mathbf{H}^T \left(\mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R}_2 \right)^{-1} \mathbf{H}\mathbf{P} + \mathbf{Q}_o
\end{aligned}$$

- **Monotonicity with respect to the state covariance matrix**

The solution to the Riccati recursion at time $k+1$ is monotonic with to the solution at time k, i.e., if $\mathbf{P}^{(1)}_k$ and $\mathbf{P}^{(2)}_k$ are two different solutions to the same Riccati recursion at time k, with $\mathbf{P}^{(1)}_k \succeq \mathbf{P}^{(2)}_k$ then $\mathbf{P}^{(1)}_{k+1} \succeq \mathbf{P}^{(2)}_{k+1}$. In order to prove the result in the general case, in which $\mathbf{P}^{(1)}_k$ and $\mathbf{P}^{(2)}_k$ are positive semidefinite, we use the following expression that relates the one-step ahead solutions to two Riccati recursions with identical $\mathbf{H}$, $\mathbf{Q}$ and $\mathbf{R}$ matrices, but different initial values $\mathbf{P}^{(1)}_k$ and $\mathbf{P}^{(2)}_k$ ([17]). It is

$$\mathbf{P}^{(2)}_{k+1} - \mathbf{P}^{(1)}_{k+1} = F_{p,k} \left(\left(\mathbf{P}^{(2)}_k - \mathbf{P}^{(1)}_k \right) - \left(\mathbf{P}^{(2)}_k - \mathbf{P}^{(1)}_k \right) \mathbf{H}^T \left(\mathbf{H}\mathbf{P}^{(2)}_k \mathbf{H}^T + \mathbf{R} \right) \mathbf{H} \left(\mathbf{P}^{(2)}_k - \mathbf{P}^{(1)}_k \right) \right) F^T_{p,k} \tag{72}$$

where $F_{p,k}$ is a matrix whose exact structure is not important for the purposes of this proof. Since we have assumed $\mathbf{P}^{(1)}_k \succeq \mathbf{P}^{(2)}_k$ we can write $\mathbf{P}^{(2)}_k - \mathbf{P}^{(1)}_k \preceq 0$. Additionally, the matrix

$$\left(\mathbf{P}^{(2)}_k - \mathbf{P}^{(1)}_k \right) \mathbf{H}^T \left(\mathbf{H}\mathbf{P}^{(2)}_k \mathbf{H}^T + \mathbf{R} \right) \mathbf{H} \left(\mathbf{P}^{(2)}_k - \mathbf{P}^{(1)}_k \right)$$

is positive semidefinite, and therefore we have

$$\begin{aligned}
-\left(\mathbf{P}^{(2)}_k - \mathbf{P}^{(1)}_k \right) \mathbf{H}^T \left(\mathbf{H}\mathbf{P}^{(2)}_k \mathbf{H}^T + \mathbf{R} \right) \mathbf{H} \left(\mathbf{P}^{(2)}_k - \mathbf{P}^{(1)}_k \right) &\preceq 0 \Rightarrow \\
\left(\mathbf{P}^{(2)}_k - \mathbf{P}^{(1)}_k \right) - \left(\mathbf{P}^{(2)}_k - \mathbf{P}^{(1)}_k \right) \mathbf{H}^T \left(\mathbf{H}\mathbf{P}^{(2)}_k \mathbf{H}^T + \mathbf{R} \right) \mathbf{H} \left(\mathbf{P}^{(2)}_k - \mathbf{P}^{(1)}_k \right) &\preceq 0 \Rightarrow \\
F_{p,k} \left(\left(\mathbf{P}^{(2)}_k - \mathbf{P}^{(1)}_k \right) - \left(\mathbf{P}^{(2)}_k - \mathbf{P}^{(1)}_k \right) \mathbf{H}^T \left(\mathbf{H}\mathbf{P}^{(2)}_k \mathbf{H}^T + \mathbf{R} \right) \mathbf{H} \left(\mathbf{P}^{(2)}_k - \mathbf{P}^{(1)}_k \right) \right) F^T_{p,k} &\preceq 0 \Rightarrow \\
\mathbf{P}^{(2)}_{k+1} - \mathbf{P}^{(1)}_{k+1} &\preceq 0
\end{aligned}$$

The last line implies that $\mathbf{P}_{k+1}^{(1)} \succeq \mathbf{P}_{k+1}^{(2)}$, which is the desired result.

We can now employ induction to prove the main statement of this appendix. Assuming that at some time instant i, $\mathbf{P}_i^u \succeq \mathbf{P}_i$, we can write

$$\begin{aligned}
\mathbf{P}_{i+1}^u &= \mathbf{P}_i^u - \mathbf{P}_i^u \mathbf{H}_o'^T \left(\mathbf{H}_o' \mathbf{P}_i^u \mathbf{H}_o'^T + \mathbf{R}_u' \right)^{-1} \mathbf{H}_o' \mathbf{P}_i^u + \mathbf{G}_o \mathbf{Q}_u \mathbf{G}_o^T \\
&\succeq \mathbf{P}_i - \mathbf{P}_i \mathbf{H}_o'^T \left(\mathbf{H}_o' \mathbf{P}_i \mathbf{H}_o'^T + \mathbf{R}_u' \right)^{-1} \mathbf{H}_o' \mathbf{P}_i + \mathbf{G}_o \mathbf{Q}_u \mathbf{G}_o^T \\
&\succeq \mathbf{P}_i - \mathbf{P}_i \mathbf{H}_o'^T \left(\mathbf{H}_o' \mathbf{P}_i \mathbf{H}_o'^T + \mathbf{R}_u' \right)^{-1} \mathbf{H}_o' \mathbf{P}_i + \mathbf{G}_o \mathbf{Q}_r(k+1) \mathbf{G}_o^T \\
&\succeq \mathbf{P}_i - \mathbf{P}_i \mathbf{H}_o'^T \left(\mathbf{H}_o' \mathbf{P}_i \mathbf{H}_o'^T + \mathbf{R}_o'(k+1) \right)^{-1} \mathbf{H}_o' \mathbf{P}_i + \mathbf{G}_o \mathbf{Q}_r(k+1) \mathbf{G}_o^T = \mathbf{P}_{i+1}
\end{aligned}$$

where the monotonicity of the Riccati recursion with respect to the covariance matrix, the property $\mathbf{Q}_u \succeq \mathbf{Q}_r(k+1)$ and the monotonicity of the Riccati recursion with respect to the measurement covariance matrix have been used in the last three lines. Thus $\mathbf{P}_i^u \succeq \mathbf{P}_i \Rightarrow \mathbf{P}_{i+1}^u \succeq \mathbf{P}_{i+1}$. For $i = 0$ the condition $\mathbf{P}_i^u \succeq \mathbf{P}_i$ holds with equality, and therefore for any $i > 0$, the solution to the Riccati recursion in Eq. (69) is an upper bound to the solution of the recursion in Eq. (70).

B Riccati Recursion for the Upper Bound on the Average Covariance

In this appendix we prove that if $\bar{\mathbf{R}}'$ and $\bar{\mathbf{Q}}_r$ are matrices such that $\bar{\mathbf{R}}' = E\{\mathbf{R}_o'(k)\}$ and $\bar{\mathbf{Q}}_r = \{\mathbf{Q}_r(k)\}$ for all $k \geq 0$, then the solutions to the following two Riccati recursions

$$\mathbf{P}_{k+1} = \mathbf{P}_k - \mathbf{P}_k \mathbf{H}_o'^T \left(\mathbf{H}_o' \mathbf{P}_k \mathbf{H}_o'^T + \mathbf{R}_o'(k+1) \right)^{-1} \mathbf{H}_o' \mathbf{P}_k + \mathbf{G}_o \mathbf{Q}_r(k+1) \mathbf{G}_o \tag{73}$$

and

$$\bar{\mathbf{P}}_{k+1} = \bar{\mathbf{P}}_k - \bar{\mathbf{P}}_k \mathbf{H}_o'^T \left(\mathbf{H}_o' \bar{\mathbf{P}}_k \mathbf{H}_o'^T + \bar{\mathbf{R}}' \right)^{-1} \mathbf{H}_o' \bar{\mathbf{P}}_k + \mathbf{G}_o \bar{\mathbf{Q}}_r \mathbf{G}_o^T \tag{74}$$

with the *same* initial condition, $\mathbf{P}_0$, satisfy $\bar{\mathbf{P}}_k \succeq E\{\mathbf{P}_k\}$ for all $k \geq 0$. We first prove a useful intermediate result:

- **Concavity of the Riccati recursion**

We note that the Riccati recursion

$$P_{k+1} = P_k - P_k H^T \left(H P_k H^T + R_{k+1} \right)^{-1} H P_k + G Q_{k+1} G \tag{75}$$

can equivalently be written as

$$\begin{aligned}
P_{k+1} = &\begin{bmatrix} I & \mathbf{0} \end{bmatrix} \begin{bmatrix} P_k & \mathbf{0} \\ \mathbf{0} & R_{k+1} \end{bmatrix} \begin{bmatrix} I \\ \mathbf{0} \end{bmatrix} \\
&- \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \begin{bmatrix} P_k & \mathbf{0} \\ \mathbf{0} & R_{k+1} \end{bmatrix} \begin{bmatrix} H^T \\ \mathbf{0} \end{bmatrix} \left(\begin{bmatrix} H & I \end{bmatrix} \begin{bmatrix} P_k & \mathbf{0} \\ \mathbf{0} & R_{k+1} \end{bmatrix} \begin{bmatrix} H^T \\ \mathbf{I} \end{bmatrix} \right)^{-1} \begin{bmatrix} H & \mathbf{0} \end{bmatrix} \begin{bmatrix} P_k & \mathbf{0} \\ \mathbf{0} & R_{k+1} \end{bmatrix} \begin{bmatrix} I \\ \mathbf{0} \end{bmatrix} \\
&+ G Q_{k+1} G
\end{aligned}$$

our goal is to show that the above expression is concave with respect to the matrix

$$\begin{bmatrix} P_k & \mathbf{0} \\ \mathbf{0} & R_{k+1} \end{bmatrix}$$

A sufficient condition for this is that the function

$$f(X) = AXB \left(CXC^T \right)^{-1} B^T X A^T \tag{76}$$

is convex with respect to the positive semidefinite matrix X, when A,B,C are arbitrary matrices of compatible dimensions. This is equivalent to proving the convexity of the function of the scalar variable t

$$f_t(t) = A(X_o + tZ_o)B\left(C(X_o + tZ_o)C^T\right)^{-1} B^T(X_o + tZ_o)A^T \tag{77}$$

with domain those values of t for which $X_o + tZ_o \succeq 0, X_o \succeq 0$ is convex [18]. $f_t(t)$ is convex if and only if the scalar function

$$f_t'(t) = z^T A(X_o + tZ_o)B\left(C(X_o + tZ_o)C^T\right)^{-1} B^T(X_o + tZ_o)A^T z \tag{78}$$

is convex for any vector z of appropriate dimensions [18]. Moreover, it is well known that a function is convex if and only if its epigraph is a convex set, and therefore we obtain the following convexity condition for $f(X)$:

$$f(X) \text{ convex } \Leftrightarrow \ \{s,t|z^T A(X_o + tZ_o)B\left(C(X_o + tZ_o)C^T\right)^{-1} B^T(X_o + tZ_o)A^T z \leq s\} \text{ is convex}$$

However, from the properties of Schur complements it is well known that if $A_o \succ 0$ then

$$\begin{bmatrix} A_o & B_o \\ B_o^T & C_o \end{bmatrix} \succeq 0 \Leftrightarrow C_o - B_o^T C_o^{-1} B \succeq 0$$

In our problem, the matrix $C(X_o + tZ_o)C^T$ is clearly positive definite, and thus we can write

$$z^T A(X_o{+}tZ_o)B\left(C(X_o + tZ_o)C^T\right)^{-1} B^T(X_o{+}tZ_o)A^T z \leq s \Leftrightarrow \begin{bmatrix} C(X_o + tZ_o)C^T & B^T(X_o + tZ_o)A^T z \\ z^T A(X_o + tZ_o)B & s \end{bmatrix} \succeq 0$$

However, the defining matrix inequality of the epigraph is equivalent to

$$\begin{bmatrix} CX_oC^T & B^T X_o A^T z \\ z^T A X_o B & 0 \end{bmatrix} + t\begin{bmatrix} CZ_oC^T & B^T Z_o A^T z \\ z^T A Z_o B & 0 \end{bmatrix} + s\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \succeq 0$$

which defines a convex set in (s,t) [18].

Thus, by the preceding analysis $f(X)$ is a convex function, and consequently P_{k+1} is a concave function of the matrix

$$\begin{bmatrix} P_k & \mathbf{0} \\ \mathbf{0} & R_{k+1} \end{bmatrix}$$

■

We now employ this result to prove the main result of this appendix. The proof is carried out by induction. Assuming that at time step k the inequality $\bar{\mathbf{P}}_k \succeq E\{\mathbf{P}_k\}$ holds, we will show that it also holds for the time step $k+1$. We have

$$\begin{aligned}
\mathbf{P}_{k+1} &= \mathbf{P}_k - \mathbf{P}_k\mathbf{H}_o'^T\left(\mathbf{H}_o'\mathbf{P}_k\mathbf{H}_o'^T + \mathbf{R}_o'(k+1)\right)^{-1}\mathbf{H}_o'\mathbf{P}_k + \mathbf{G}_o\mathbf{Q}(k+1)\mathbf{G}_o^T \Rightarrow \\
E\{\mathbf{P}_{k+1}\} &= E\{\mathbf{P}_k - \mathbf{P}_k\mathbf{H}_o'^T\left(\mathbf{H}_o'\mathbf{P}_k\mathbf{H}_o'^T + \mathbf{R}_o'(k+1)\right)^{-1}\mathbf{H}_o'\mathbf{P}_k + \mathbf{G}_o\mathbf{Q}(k+1)\mathbf{G}_o^T\} \\
&= E\{\mathbf{P}_k - \mathbf{P}_k\mathbf{H}_o'^T\left(\mathbf{H}_o'\mathbf{P}_k\mathbf{H}_o'^T + \mathbf{R}_o'(k+1)\right)^{-1}\mathbf{H}_o'\mathbf{P}_k\} + \mathbf{G}_oE\{\mathbf{Q}(k+1)\}\mathbf{G}_o^T \\
&\preceq E\{\mathbf{P}_k\} - E\{\mathbf{P}_k\}\mathbf{H}_o'^T\left(\mathbf{H}_o'E\{\mathbf{P}_k\}\mathbf{H}_o'^T + E\{\mathbf{R}_o'(k+1)\}\right)^{-1}\mathbf{H}_o'E\{\mathbf{P}_k\} + \mathbf{G}_oE\{\mathbf{Q}(k+1)\}\mathbf{G}_o^T
\end{aligned}$$

where in the last line the concavity of Jensen's inequality was applied [18], in order to exploit tht concavity of the Riccati. By assumption, $\bar{\mathbf{P}}_k \succeq E\{\mathbf{P}_k\}$ and employing the property of the monotonicity of the Riccati with respect to the covariance matrix (cf. Appendix A), we can write

$$\begin{aligned}
E\{\mathbf{P}_{k+1}\} &\preceq \bar{\mathbf{P}}_k - \bar{\mathbf{P}}_k\mathbf{H}_o'^T\left(\mathbf{H}_o'\bar{\mathbf{P}}_k\mathbf{H}_o'^T + E\{\mathbf{R}_o'(k+1)\}\right)^{-1}\mathbf{H}_o'\bar{\mathbf{P}}_k + \mathbf{G}_oE\{\mathbf{Q}(k+1)\}\mathbf{G}_o^T \\
&= \bar{\mathbf{P}}_k - \bar{\mathbf{P}}_k\mathbf{H}_o'^T\left(\mathbf{H}_o'\bar{\mathbf{P}}_k\mathbf{H}_o'^T + \bar{\mathbf{R}}'\}\right)^{-1}\mathbf{H}_o'\bar{\mathbf{P}}_k + \mathbf{G}_o\bar{\mathbf{Q}}_r\mathbf{G}_o^T \\
&= \bar{\mathbf{P}}_{k+1}
\end{aligned}$$

Thus, $\bar{\mathbf{P}}_k \succeq E\{\mathbf{P}_k\} \Rightarrow \bar{\mathbf{P}}_{k+1} \succeq E\{\mathbf{P}_{k+1}\}$. For $k=0$ the condition $\bar{\mathbf{P}}_k \succeq E\{\mathbf{P}_k\}$ holds with equality, and the proof is complete.

C Matrix Inversion Lemma

If A is $n \times n$, B is $n \times m$, C is $m \times m$ and D is $m \times n$ then:

$$(A^{-1} + BC^{-1}D)^{-1} = A - AB(DAB + C)^{-1}DA \tag{79}$$

D Rank and Nullspace of the Measurement Matrices

In this appendix we present some results concerning the rank of the measurement matrices in CL, as well as the rank and eigenvectors of the matrix:

$$\mathbf{C}_s = \mathbf{Q}_s^{1/2}\mathbf{H}_o^T\mathbf{R}_s^{-1}\mathbf{H}_o\mathbf{Q}_s^{1/2}$$

Where the matrices $\mathbf{Q}_s^{1/2}$ and $\mathbf{R}_s$ can be substituted for either by the upper bounds, or by the average values of the corresponding covariance matrices.

We first note that,in the case in which the robots receive only relative position measurements, $\mathbf{H}_o$ consists of block rows of the form

$$\begin{bmatrix} \mathbf{0}_{2\times2} & .. & -I_{2\times2} & .. & I_{2\times2} & .. & \mathbf{0}_{2\times2} \end{bmatrix} = \begin{bmatrix} 0 & .. & -1 & .. & 1 & .. & 0 \end{bmatrix} \otimes I_{2\times2}$$

while if the some of the robots additionally receive absolute position measurements, $\mathbf{H}_o$ also has some block rows of the form

$$\begin{bmatrix} \mathbf{0}_{2\times2} & .. & I_{2\times2} & .. & \mathbf{0}_{2\times2} \end{bmatrix} = \begin{bmatrix} 0 & .. & 1 & .. & 0 \end{bmatrix} \otimes I_{2\times2}$$

We therefore conclude, that in any case, the matrix $\mathbf{H}_o$ can be expressed as

$$\mathbf{H}_o = H \otimes I_{2\times2} \tag{80}$$

where H is an appropriate matrix, consisting of rows having one of the two following structures:

$$H_{ij} = \begin{bmatrix} 0 & .. & -1 & .. & 1 & .. & 0 \end{bmatrix}$$

or

$$H_\ell = \begin{bmatrix} 0 & .. & 1 & .. & 0 \end{bmatrix}$$

It becomes clear that the matrix H will be the measurement matrix associated with a 1D CL system model, in which the robot team has the same RPMG as the team of robots performing localization in 2D (cf. Section **??**).

Employing the properties of the Kronecker product, from Eq. (80) we conclude that

$$\text{rank}(\mathbf{H}_o) = \text{rank}(H)\,\text{rank}(I_{2\times2}) = 2 \cdot \text{rank}(H) \tag{81}$$

and therefore we can determine the rank of $\mathbf{H}_o$ by first studying the properties of the 1D-measurement matrix H. For this reason, we start by presenting the results for the, simpler, one-dimensional CL case (cf. Section **??**).

D.1 Cooperative Localization in 1D

For the one-dimensional case, when no absolute position measurements are available, the measurement matrix H is defined by

$$H = \begin{bmatrix} H_{ij} \\ \vdots \\ H_{kl} \\ \vdots \\ H_{mn} \end{bmatrix} \tag{82}$$

where each row of H corresponds to one relative position measurement, or equivalently, to one edge of the RPMG. Each of the rows contains a “-1”, at the column that corresponds to the robot i registering the relative position measurement, and a “1” at the column that corresponds to the robot being observed. This matrix is identical to the *incidence* *matrix* defined for any directed graph. In [19] it is shown that the incidence matrix of a directed graph is of rank $N-1$, whenever the graph is connected, and therefore the rank of H is $N-1$, where we have imposed the constraint that the measurement graph is connected[3].

Having determined the rank of H, we are now able to study the rank and eigenvectors of the matrix

$$C = Q^{1/2} H^T R^{-1} H Q^{1/2}$$

where Q and R are diagonal and positive definite. In order to determine the rank of this matrix, we use the following lemma from linear algebra [19]:

Lemma D.1 *The rank of the product of two matrices A, B is given by*

$$\text{rank}(AB) = \text{rank}(B) - \dim \left(N(A) \bigcap R(B) \right) \tag{83}$$

where $\dim\ X \bigcap Y$ *denotes the dimension of the subspace formed by the intersection of the subspaces* X *and* Y*,* $N(A)$ *is the nullspace of matrix A, and* $R(B)$ *is the range of B.*

Note that the matrix product $H^T R^{-1} H$ can be written as $H^T R^{-1/2} R^{-1/2} H = (R^{-1/2} H)^T R^{-1/2} H$. We now apply the above lemma to the matrix product $M = R^{-1/2} H$. Since $R^{-1/2}$ is an invertible matrix, its nullspace is of dimension 0, and we have $\text{rank}(M) = \text{rank}(R^{-1/2} H) = \text{rank}(H) = N-1$. Moreover, it is evident that the nullspace of M will be the same with the nullspace of H. In order to find the rank of $H^T R^{-1} H = M^T M$ we employ the above lemma once again:

$$\text{rank}(H^T R^{-1} H) = \text{rank}(M^T M) = \text{rank}(M) - \dim \left(N(M) \bigcap R(M) \right)$$

Since the nullspace and the range of any matrix are disjoint sets, $\text{rank}(H^T R^{-1} H) = N-1$. By consecutive application of the above lemma to the matrix products $(H^T R^{-1} H) Q^{1/2}$ and $Q^{1/2}(H^T R^{-1} H Q^{1/2})$ it is easy to show that $\text{rank}(C) = N-1$.

A direct consequence of this result is that C has one eigenvalue equal to zero, and that its nullspace is of dimension 1. Note that since the sum of all elements of the rows of H is zero, we obtain

$$H \mathbf{1}_{N\times 1} = \mathbf{0}_{N\times 1}$$

hence the basis of the nullspace of H is the vector $x_N = \mathbf{1}_{N\times 1}$. As a result, we deduce that the basis vector for the nullspace of C is given by

$$U_N = \frac{1}{||Q^{-1/2}\mathbf{1}_{N\times 1}||} Q^{-1/2} \mathbf{1}_{N\times 1}$$

since

$$CU_N = \frac{1}{||Q^{-1/2}\mathbf{1}_{N\times 1}||} Q^{1/2} H^T R^{-1} H Q^{1/2} Q^{-1/2} \mathbf{1}_{N\times 1} = \frac{1}{||Q^{-1/2}\mathbf{1}_{N\times 1}||} Q^{1/2} H^T R^{-1} \left(H \mathbf{1}_{N\times 1} \right) = \mathbf{0}_{N\times 1}$$

Simpe calculations show that

$$U_N = \frac{1}{||Q^{-1/2}\mathbf{1}_{N\times 1}||} Q^{-1/2} \mathbf{1}_{N\times 1} = \frac{1}{\left(\sum_{i=1}^{N} \frac{1}{q_i} \right)^{1/2}} Q^{-1/2} \mathbf{1}_{N\times 1} = \sqrt{q_{total}}\, Q^{-1/2} \mathbf{1}_{N\times 1}$$

where

$$\frac{1}{q_{total}} = \sum_{i=1}^{N} \frac{1}{q_i}$$

Finally, by applying simple vector-matrix multiplication, we obtain the following result, which is useful in the derivations in Section **??**:

$$Q^{1/2} U_N U_N^T Q^{1/2} = q_{total} \mathbf{1}_{N\times N} \tag{84}$$

[3]This is not a restraining assumption. The case in which the RPMG is not connected is a degenerate one. In this case, the robots that are not connected by an edge to any robot of the team, do not actually belong to the team, and therefore, we can study this case by a considering each connected subgraph independently.

If in addition to the relative position measurements, some of the robots receive absolute positioning information, then the measurement matrix has a number of rows (at least one) of the form $H_{i_A} = [0 \; .. \; 1 \; .. \; 0]$, with the "1"s being at the columns corresponding to the robots receiving absolute positioning information. In this case C can be written as

$$C = Q^{1/2}\left(H^T R^{-1} H + \sum_k \frac{1}{\sigma_A^2} H_{k_A}^T H_{k_A}\right) Q^{1/2} = C + Q^{1/2} \sum_k \frac{1}{\sigma_{A_k}^2} H_{k_A}^T H_{k_A} Q^{1/2} = C + C_A \qquad (85)$$

where the sum is over all robots receiving absolute position measurements, $\sigma_{A_k}^2$ are the variances of these measurements, and C is the matrix of the previous case, in which only relative position information were available.

We now prove that C is positive definite, by showing that $x^T C x = 0 \Leftrightarrow x = 0$. Assume that there exists a vector x such that

$$x^T C x = 0 \Rightarrow x^T C x + x^T C_A x = 0$$

Clearly, both terms in the last expression are always nonnegative, since the involved matrices are positive semidefinite. Thus $x^T C x = 0$ implies $x^T C x = x^T C_A x = 0$. The term $x^T C x$ assumes the zero value only when $x = aU_N$, where $a \in \mathbf{R}$ and U_N is the basis vector of the nullspace of C. But

$$a^2 U_N^T \left(Q^{1/2} \sum_k \frac{1}{\sigma_{A_k}^2} H_{k_A}^T H_{k_A} Q^{1/2}\right) U_N = a^2 q_{total} \sum_k \frac{1}{\sigma_{A_k}^2}$$

and therefore this quantity is equal to zero only when $a = 0$. Thus $x^T C x = 0 \Rightarrow x = 0$, which implies that when at least one robot has access to absolute position information, C is positive definite.

D.2 Cooperative Localization in 2D

We can now employ the results of the preceding 1D analysis to the 2D case. Using the result of Eq. (81), we immediately see that when the robots of the a team performing CL in 2D only record relative position measurements, then $\mathrm{rank}(\mathbf{H}_o) = 2N - 2$, while if at least one of the robots has access to absolute position measurements, we have $\mathrm{rank}(\mathbf{H}_o) = 2N$.

Regarding the rank and eigenvectors of $\mathbf{C}_s$, it is straightforward to see that

$$\mathrm{rank}(\mathbf{H}_o) = 2N \Rightarrow \mathrm{rank}(\mathbf{C}_s) = 2N$$

since in this case $\mathbf{C}_s$ is the product of full-rank matrices. Similarly, we can use Lemma D.1 in the same way as in the 1D case, to show that $\mathrm{rank}(\mathbf{C}_s) = 2N - 2$. As a result, the nullspace of $\mathbf{C}_s$ is of dimension 2, and is spanned by 2 orthogonal basis vectors. We can find two such vectors by observing that

$$\begin{aligned}
\mathbf{C}_s\left(\mathbf{Q}_s^{-1/2}\mathbf{1}_{N\times 1} \otimes I_{2\times 2}\right) &= \mathbf{Q}_s^{1/2}\mathbf{H}_o^T\mathbf{R}_s^{-1}\mathbf{H}_o\mathbf{Q}_s^{1/2}\left(\mathbf{Q}_s^{-1/2}\mathbf{1}_{N\times 1} \otimes I_{2\times 2}\right) \\
&= \mathbf{Q}_s^{1/2}\mathbf{H}_o^T\mathbf{R}_s^{-1}\mathbf{H}_o\left(\mathbf{1}_{N\times 1} \otimes I_{2\times 2}\right) \\
&= \mathbf{Q}_s^{1/2}\mathbf{H}_o^T\mathbf{R}_s^{-1}\left(H \otimes I_{2\times 2}\right)\left(\mathbf{1}_{N\times 1} \otimes I_{2\times 2}\right)
\end{aligned}$$

But employing the properties of the Kronecker product yields

$$\left(H \otimes I_{2\times 2}\right)\left(\mathbf{1}_{N\times 1} \otimes I_{2\times 2}\right) = \left(H\mathbf{1}_{N\times 1}\right) \otimes I_{2\times 2} = \mathbf{0}_{2N\times 2}$$

and therefore

$$\mathbf{C}_s\left(\mathbf{Q}_s^{-1/2}\mathbf{1}_{N\times 1} \otimes I_{2\times 2}\right) = \mathbf{0}_{2N\times 2}$$

The columns of the matrix $\mathbf{Q}_s^{-1/2}\mathbf{1}_{N\times 1} \otimes I_{2\times 2}$ are

$$c_1 = \mathbf{Q}_s^{-1/2}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}$$

and

$$c_1 = \mathbf{Q}_s^{-1/2} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ \vdots \end{bmatrix}$$

which are orthogonal (this is easily verified by computing the dot product $c_1^T c_2$). Therefore, a basis for the nullspace of $\mathbf{C}_s$ is given by the vectors

$$\mathbf{U}_{2N-1} = \frac{c_1}{||c_1||} = \sqrt{q_{s_T}} \mathbf{Q}_s^{-1/2} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} \tag{86}$$

and

$$\mathbf{U}_{2N} = \frac{c_2}{||c_2||} = \sqrt{q_{s_T}} \mathbf{Q}_s^{-1/2} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ \vdots \end{bmatrix} \tag{87}$$

E Matrix Monotonicity of M_N

In this appendix we show that the matrix

$$M \quad = \quad V^T \mathbf{X} \left(I_{2N \times 2N} + h(\mathbf{C}_s) \mathbf{X} \right)^{-1} V \tag{88}$$

is matrix increasing in the argument $\mathbf{X}$, i.e.,

$$\mathbf{X}' \succeq \mathbf{X} \Rightarrow M' \succeq M \tag{89}$$

We note that if $\mathbf{X}$ is invertible (which is the case of interest), then

$$M \quad = \quad V^T \left(\mathbf{X}^{-1} + h(\mathbf{C}_s) \right)^{-1} V \tag{90}$$

And from the last relation it follows that

$$\begin{aligned}
\mathbf{X}' &\succeq \mathbf{X} \Rightarrow \\
\mathbf{X}'^{-1} &\preceq \mathbf{X}^{-1} \Rightarrow \\
\mathbf{X}'^{-1} + h(\mathbf{C}_s) &\preceq \mathbf{X}^{-1} + h(\mathbf{C}_s) \Rightarrow \\
\left(\mathbf{X}'^{-1} + h(\mathbf{C}_s) \right)^{-1} &\succeq \left(\mathbf{X}^{-1} + h(\mathbf{C}_s) \right)^{-1} \Rightarrow \\
V^T \left(\mathbf{X}'^{-1} + h(\mathbf{C}_s) \right)^{-1} V &\succeq V^T \left(\mathbf{X}^{-1} + h(\mathbf{C}_s) \right)^{-1} V \Rightarrow \\
M' &\succeq M
\end{aligned}$$

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