IMU-Camera Calibration: Observability Analysis

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1 Introduction

Attitude estimation and filtering is of essential importance in today's vehicular and navigational technologies. The core in most of the algorithms relies on the integration of inertial measurements, namely linear accelerations and rotational velocity. That, however, introduces the well known drift problem which is basically a result of integrating the error in the measurements along with the measurements themselves. Additionally, existence of unknown biases in the inertial measurements complicates the problem even more by introducing correlation between measurements in different time-steps.

A well known solution for the correlation between inertial measurements is inclusion of the biases in the system model and consequently estimation of them along with the attitude, velocity and position [1]. Implementation of an Extended Kalman Filter (EKF) with this system model and employment of some external measurements, such as Global Position System (GPS) measurements can resolve the drift problem.

Recent advances in manufacturing of low cost cameras and weaknesses in GPS measurements, has created a tendency to use visual information as the sole external measurements available to the EKF. However, proper fusion of the camera measurement and inertial measurements requires a precise knowledge of the physical transformation (rotation and translation) between the camera and the Inertial Measurement Unit (IMU). An algorithm for estimation of this transformation is developed in [2] where 6 parameters describing the translation and rotation between the IMU and the camera are put in the state vector and the measurements of the known visual landmarks are used to update the corresponding EKF.

This technical report investigates the observability of the system proposed in [2] using Lie derivatives and the observability rank condition [3].

2 Problem Formulation

In this report, we study the observability of EKF estimator employed for determining the IMU-camera calibration parameters simultaneously with the IMU pose, velocity, and biases. First, to simplify notation, the components of the filter state at time t are renamed as

$$\mathbf{x}(t) = \left[{}^I_G\hat{\bar{q}}^T(t)\ \ \hat{\mathbf{b}}_g^T(t)\ \ {}^G\hat{\mathbf{v}}_I^T(t)\ \ \hat{\mathbf{b}}_a^T(t)\ \ {}^G\hat{\mathbf{p}}_I^T(t)\ \ {}^I_C\hat{\bar{q}}^T(t)\ \ {}^I\hat{\mathbf{p}}_C^T(t)\right]^T = \left[\bar{q}_I^T\ \ \mathbf{b}_g^T\ \ \mathbf{v}^T\ \ \mathbf{b}_a^T\ \ \mathbf{p}_I^T\ \ \bar{q}_C^T\ \ \mathbf{p}_C^T\right]^T \tag{1}$$

where the time argument is not repeated to reduce the complexity of the expressions. We recall the kinematic equations that describe the system's motion [1, 2]:

$$\dot{\bar{q}}_I = \frac{1}{2}\boldsymbol{\Omega}(\boldsymbol{\omega}_m - \mathbf{b}_g)\bar{q}_I = \frac{1}{2}\boldsymbol{\Xi}(\bar{q}_I)(\boldsymbol{\omega}_m - \mathbf{b}_g) \tag{2}$$

$$\dot{\mathbf{p}}_I = \mathbf{v}_I \quad , \quad \dot{\mathbf{v}}_I = \mathbf{a}_m - \mathbf{b}_a \tag{3}$$
$$\dot{\mathbf{b}}_g = \mathbf{0}_{3\times 1} \quad , \quad \dot{\mathbf{b}}_a = \mathbf{0}_{3\times 1} \tag{4}$$
$$\dot{\bar{q}}_C = \mathbf{0}_{3\times 1} \quad , \quad \dot{\mathbf{p}}_C = \mathbf{0}_{3\times 1} \tag{5}$$

In these expressions $\boldsymbol{\omega}_m = [\omega_x \ \ \omega_y \ \ \omega_z]^T$ is the measured rotational velocity in the IMU frame, $\mathbf{a}_m$ is the measured linear acceleration in the global frame, and

$$\boldsymbol{\Omega}(\boldsymbol{\omega}) = \begin{bmatrix} -\lfloor \boldsymbol{\omega} \times \rfloor & \boldsymbol{\omega} \\ -\boldsymbol{\omega}^T & 0 \end{bmatrix} \quad , \quad \boldsymbol{\Xi}(\bar{q}) = \begin{bmatrix} q_4 \mathbf{I}_{3\times 3} + \lfloor \mathbf{q} \times \rfloor \\ -\mathbf{q}^T \end{bmatrix} \tag{6}$$

where the skew-symmetric matrix $\lfloor \boldsymbol{\omega} \times \rfloor$ is defined as

$$\lfloor \boldsymbol{\omega} \times \rfloor = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}.$$

We rearrange the nonlinear kinematic equations (2)-(5) in a suitable format for computing the Lie derivatives where $\boldsymbol{\omega}_m$ and $\mathbf{a}_m$ are treated as control inputs:

$$\begin{bmatrix} \dot{\bar{q}}_I \\ \dot{\mathbf{b}}_g \\ \dot{\mathbf{v}} \\ \dot{\mathbf{b}}_a \\ \dot{\mathbf{p}}_I \\ \dot{\bar{q}}_C \\ \dot{\mathbf{p}}_C \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{1}{2}\boldsymbol{\Xi}(\bar{q}_I)\mathbf{b}_g \\ \mathbf{0}_{3\times 1} \\ \mathbf{g} - \mathbf{C}^T(\bar{q}_I)\mathbf{b}_a \\ \mathbf{0}_{3\times 1} \\ \mathbf{v} \\ \mathbf{0}_{3\times 1} \\ \mathbf{0}_{3\times 1} \end{bmatrix}}_{f_0} + \underbrace{\begin{bmatrix} \frac{1}{2}\boldsymbol{\Xi}(\bar{q}_I) \\ \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} \end{bmatrix}}_{\mathbf{f}_1} \boldsymbol{\omega}_m + \underbrace{\begin{bmatrix} \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} \\ \mathbf{C}^T(\bar{q}_I) \\ \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} \end{bmatrix}}_{\mathbf{f}_2} \mathbf{a}_m \tag{7}$$

As the IMU-camera moves, it records images of a calibration pattern. These are then processed to detect and identify point features whose positions ${}^G\mathbf{p}_{f_i}$ are known with respect to the global frame of reference. A least-squares corner extractor is used for feature detection and data association. Once this process is completed for each image, a list of point features along with their measured image coordinates $(u_i, \ v_i)$ is provided to the EKF which uses them to update the state estimates. We hereafter consider the case when 3 or more features are detected in each calibration image processed by the filter. As it is shown in [4] the *camera pose* is observable in this case. For the purpose of observability analysis, we use this fact to rewrite the measurement equation as follows:

$$\mathbf{z}_1 = \mathbf{h}_1(\mathbf{x}) + \eta_1 = {}^G_C\bar{q} + \eta_1 = \bar{q}_I^{-1} \otimes \bar{q}_C + \eta_1 = \mathbf{J}\bar{q}_I \otimes \bar{q}_C + \eta_1 \tag{8}$$
$$\mathbf{z}_2 = \mathbf{h}_2(\mathbf{x}) + \eta_2 = {}^G\mathbf{p}_C + \eta_2 = \mathbf{p}_I + \mathbf{C}^T(\bar{q}_I)\mathbf{p}_C + \eta_2 \tag{9}$$

where $\mathbf{C}(\bar{q})$ is the rotational matrix corresponding to quaternion $\bar{q}$, $\otimes$ denotes quaternion multiplication, and,

$$\bar{q}^{-1} = \mathbf{J}\bar{q}, \quad \mathbf{J} = \begin{bmatrix} -\mathbf{I}_{3\times 3} & 0 \\ 0 & 1 \end{bmatrix} \tag{10}$$

Furthermore, we enforce the norm of the quaternions to unit by employing the following additional measurement equation:

$$h_3(\mathbf{x}) = \bar{q}_I^T \bar{q}_I - 1 = 0 \tag{11}$$
$$h_4(\mathbf{x}) = \bar{q}_C^T \bar{q}_C - 1 = 0 \tag{12}$$

In order to prove that that the system described by Eq. (7) is observable, we employ an algebraic test to show that the observability rank condition is met [3]. This algebraic test consists of demonstrating that the state space of the system (cf. Eq. (1)) is spanned by the gradients of the Lie derivatives of the measurement functions [5]. We start by taking the zeroth order Lie derivatives of $\mathbf{h}_1$, $\mathbf{h}_2$, and h_3 which are equal to themselves.

$$\mathfrak{L}_0 \mathbf{h}_1 = \mathbf{h}_1 = \bar{q}_I^{-1} \otimes \bar{q}_C \tag{13}$$

$$\mathfrak{L}^0\mathbf{h}_2 = \mathbf{h}_2 = \mathbf{p}_I + \mathbf{C}^T(\bar{q}_I)\mathbf{p}_C \tag{14}$$
$$\mathfrak{L}^0 h_3 = h_3 = \bar{q}_I^T\bar{q}_I - 1 \tag{15}$$

Therefore the gradients of the zeroth order Lie derivatives are exactly the same as the measurement function Jacobians:

$$\nabla\mathfrak{L}^0\mathbf{h}_1 = \begin{bmatrix}\mathcal{R}(\bar{q}_C)\mathbf{J} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathcal{L}(\mathbf{J}\bar{q}_I) & \mathbf{0}_{3\times3}\end{bmatrix} \tag{16}$$
$$\nabla\mathfrak{L}^0\mathbf{h}_2 = \begin{bmatrix}\boldsymbol{\Psi}(\bar{q}_I,\mathbf{p}_C) & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{I}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{C}^T(\bar{q}_I)\end{bmatrix} \tag{17}$$
$$\nabla\mathfrak{L}^0 h_3 = \begin{bmatrix}2\bar{q}_I^T & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3}\end{bmatrix} \tag{18}$$

In deriving (16) the following identities have been used [1]:

$$\bar{q}_1 \otimes \bar{q}_2 = \mathcal{L}(\bar{q}_1)\bar{q}_2 = \mathcal{R}(\bar{q}_2)\bar{q}_1 \tag{19}$$
$$\mathcal{L}(\bar{q}) = \begin{bmatrix} q_4\mathbf{I}_{3\times3} - \lfloor\mathbf{q}\times\rfloor & \mathbf{q} \\ -\mathbf{q}^T & q_4\end{bmatrix} \quad , \quad \mathcal{R}(\bar{q}) = \begin{bmatrix} q_4\mathbf{I}_{3\times3} + \lfloor\mathbf{q}\times\rfloor & \mathbf{q} \\ -\mathbf{q}^T & q_4\end{bmatrix} \tag{20}$$

In (17) $\boldsymbol{\Psi}(\bar{q},\mathbf{p})$ is defined as:

$$\boldsymbol{\Psi}(\bar{q},\mathbf{p}) = \frac{\partial\mathbf{C}^T(\bar{q})\mathbf{p}}{\partial\bar{q}} = \begin{bmatrix}-2q_4\lfloor\mathbf{p}\times\rfloor + 2(\lfloor\mathbf{p}\times\rfloor\lfloor\mathbf{q}\times\rfloor - 2\lfloor\mathbf{q}\times\rfloor\lfloor\mathbf{p}\times\rfloor) & 2\lfloor\mathbf{q}\times\rfloor\mathbf{p}\end{bmatrix} \tag{21}$$

In the derivation of $\boldsymbol{\Psi}$ a variation of Rodrigues rotation formula has been employed [1]:

$$\mathbf{C}^T(\bar{q}) = \mathbf{I}_{3\times3} + 2q_4\lfloor\mathbf{q}\times\rfloor + 2\lfloor\mathbf{q}\times\rfloor^2 \tag{22}$$

We continue by computing higher order Lie derivatives and taking their gradients, similar to (16) and (17), in order to show that these gradients span all 21 directions of the state space. The first order Lie derivatives of $\mathbf{h}_1$ and $\mathbf{h}_2$ with respect to f_0 can be computed as:

$$\mathfrak{L}_{f_0}\mathbf{h}_1 = \nabla\mathfrak{L}^0\mathbf{h}_1\cdot f_0 = -\frac{1}{2}\mathcal{R}(\bar{q}_C)\mathbf{J}\boldsymbol{\Xi}(\bar{q}_I)\mathbf{b}_g \tag{23}$$
$$\mathfrak{L}_{f_0}\mathbf{h}_2 = \nabla\mathfrak{L}^0\mathbf{h}_2\cdot f_0 = -\frac{1}{2}\boldsymbol{\Psi}(\bar{q}_I,\mathbf{p}_C)\Xi(\bar{q}_I)\mathbf{b}_g + \mathbf{v} \tag{24}$$

Taking their gradients:

$$\nabla\mathfrak{L}_{f_0}\mathbf{h}_1 = \begin{bmatrix}\mathbf{X}_1(\bar{q}_C,\mathbf{b}_g) & -\frac{1}{2}\mathcal{R}(\bar{q}_C)\mathbf{J}\boldsymbol{\Xi}(\bar{q}_I) & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{X}_2(\bar{q}_I,\mathbf{b}_g) & \mathbf{0}_{3\times3}\end{bmatrix} \tag{25}$$
$$\nabla\mathfrak{L}_{f_0}\mathbf{h}_2 = \begin{bmatrix}\mathbf{X}_3(\bar{q}_I,\mathbf{p}_C,\mathbf{b}_g) & \mathbf{X}_4(\bar{q}_I,\mathbf{p}_C) & \mathbf{I}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{X}_5(\bar{q}_I,\mathbf{b}_g)\end{bmatrix} \tag{26}$$

where $\mathbf{X}_i(\cdot),\ i = 1,2,\ldots$ are matrices that, irrespective of their value, will be eliminated in following derivation, and hence, they are not computed explicitly.

The next derivative that is of our interest is the first order Lie derivative of $\mathbf{h}_2$ with respect to $\mathbf{f}_1$. At this point, it should be noticed that $\mathbf{f}_1$ as defined in Eq. (7) is a compact representation of three column vectors. We can also write the associated Lie derivatives in a compact form:

$$\mathfrak{L}_{\mathbf{f}_1}\mathbf{h}_2 = \nabla\mathfrak{L}^0\mathbf{h}_2\cdot\mathbf{f}_1 = -\frac{1}{2}\boldsymbol{\Psi}(\bar{q}_I,\mathbf{p}_C)\boldsymbol{\Xi}(\bar{q}_I) \tag{27}$$

However, for taking the gradient of (27), we need to expand these Lie derivatives as column vectors. For this purpose, observe that the columns of $\boldsymbol{\Xi}(\bar{q})$ are basically signed permutations of $\bar{q}$. Let us represent the three columns of $\boldsymbol{\Xi}(\bar{q})$ with $\bar{s}_i,\ i = 1\cdots3$, then we have:

$$[\bar{s}_1\ ,\ \bar{s}_2\ ,\ \bar{s}_3] = [\boldsymbol{\Pi}_1\bar{q}\ ,\ \boldsymbol{\Pi}_2\bar{q}\ ,\ \boldsymbol{\Pi}_3\bar{q}] = \boldsymbol{\Xi}(\bar{q}) = \begin{bmatrix} q_4\mathbf{I}_{3\times3} + \lfloor\mathbf{q}\times\rfloor \\ -\mathbf{q}^T\end{bmatrix} = \begin{bmatrix} q_4 & -q_3 & q_2 \\ q_3 & q_4 & -q_1 \\ -q_2 & q_1 & q_4 \\ -q_1 & -q_2 & -q_3\end{bmatrix} \tag{28}$$

where $\mathbf{\Pi}_i,\ i = 1\cdots 3$ are the signed permutation matrices:

$$\mathbf{\Pi}_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \ , \ \mathbf{\Pi}_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \ , \ \mathbf{\Pi}_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \tag{29}$$

Consequently (27) can be expanded as

$$\mathfrak{L}_{\mathbf{f}_1}\mathbf{h}_2 = [\mathfrak{L}_{f_{11}}\mathbf{h}_1 \ , \ \mathfrak{L}_{f_{12}}\mathbf{h}_1 \ , \ \mathfrak{L}_{f_{13}}\mathbf{h}_1] \tag{30}$$

The gradient of columns of (30) can be stacked together to form the gradient of (27):

$$\nabla\mathfrak{L}_{\mathbf{f}_1}\mathbf{h}_2 = \begin{bmatrix} \nabla\mathfrak{L}_{f_{11}}\mathbf{h}_2 \\ \nabla\mathfrak{L}_{f_{12}}\mathbf{h}_2 \\ \nabla\mathfrak{L}_{f_{13}}\mathbf{h}_2 \end{bmatrix} \tag{31}$$

$$\nabla\mathfrak{L}_{f_{1i}}\mathbf{h}_2 = \begin{bmatrix}\mathbf{\Gamma}_i(\bar{q}_I, \mathbf{p}_C) & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{\Upsilon}_i(\bar{q}_I)\end{bmatrix} \tag{32}$$

$$\mathbf{\Upsilon}_i(\bar{q}_I) = -\frac{1}{2}\frac{\partial\mathbf{\Psi}(\bar{q}_I, \mathbf{p}_C)\bar{s}_i}{\partial\mathbf{p}_C} = -q_4\lfloor\mathbf{s}_i\times\rfloor + \lfloor\lfloor\mathbf{q}\times\rfloor\mathbf{s}_i\times\rfloor - 2\lfloor\mathbf{q}\times\rfloor\lfloor\mathbf{s}_i\times\rfloor - s_{i4}\lfloor\mathbf{q}\times\rfloor \tag{33}$$

$$\mathbf{\Gamma}_i(\bar{q}_I) = -\frac{1}{2}\frac{\partial\mathbf{\Psi}(\bar{q}_I, \mathbf{p}_C)\bar{s}_i}{\partial\bar{q}_I} \tag{34}$$

Finally we need to compute two higher order Lie derivatives of $\mathbf{h}_2$, namely $\mathfrak{L}_{\mathbf{f}_2}\mathfrak{L}_{f_0}\mathbf{h}_2$ and $\mathfrak{L}_{f_0}\mathfrak{L}_{f_0}\mathbf{h}_2$:

$$\mathfrak{L}_{\mathbf{f}_2}\mathfrak{L}_{f_0}\mathbf{h}_2 = \nabla\mathfrak{L}_{f_0}\mathbf{h}_2\cdot\mathbf{f}_2 = \mathbf{C}^T(\bar{q}_I) \tag{35}$$

$$\mathfrak{L}_{f_0}\mathfrak{L}_{f_0}\mathbf{h}_2 = \nabla\mathfrak{L}_{f_0}\mathbf{h}_2\cdot f_0 = -\frac{1}{2}\mathbf{X}_3(\bar{q}_I, \mathbf{p}_C, \mathbf{b}_g)\mathbf{\Xi}(\bar{q}_I)\mathbf{b}_g + \mathbf{g} - \mathbf{C}^T(\bar{q}_I)\mathbf{b}_a \tag{36}$$

Similar to (27), Eq. (35) is also representing 3 column vectors corresponding to 3 elements of control input $\mathbf{a}$. Each column can be shown as:

$$\mathfrak{L}_{f_{2i}}\mathfrak{L}_{f_0}\mathbf{h}_2 = \mathbf{C}^T(\bar{q}_I)\mathbf{e}_i \quad : \ i = 1\cdots 3 \tag{37}$$

$$\mathbf{e}_1 = [1\ \ 0\ \ 0]^T \ , \ \mathbf{e}_2 = [0\ \ 1\ \ 0]^T \ , \ \mathbf{e}_3 = [0\ \ 0\ \ 1]^T \tag{38}$$

thus their gradient can be written as:

$$\nabla\mathfrak{L}_{\mathbf{f}_2}\mathfrak{L}_{f_0}\mathbf{h}_2 = \begin{bmatrix} \nabla\mathfrak{L}_{f_{21}}\mathfrak{L}_{f_0}\mathbf{h}_2 \\ \nabla\mathfrak{L}_{f_{22}}\mathfrak{L}_{f_0}\mathbf{h}_2 \\ \nabla\mathfrak{L}_{f_{23}}\mathfrak{L}_{f_0}\mathbf{h}_2 \end{bmatrix} \ , \ \nabla\mathfrak{L}_{f_{2i}}\mathfrak{L}_{f_0}\mathbf{h}_2 = \begin{bmatrix}\mathbf{\Psi}(\bar{q}_I, \mathbf{e}_i) & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3}\end{bmatrix} \tag{39}$$

The last part is the gradient of (36) that can be computed as:

$$\nabla\mathfrak{L}_{f_0}\mathfrak{L}_{f_0}\mathbf{h}_2 = \begin{bmatrix}\mathbf{X}_6(\bar{q}_I, \mathbf{p}_C, \mathbf{b}_g, \mathbf{b}_a) & \mathbf{X}_7(\bar{q}_I, \mathbf{p}_C, \mathbf{b}_g, \mathbf{b}_a) & \mathbf{0}_{3\times3} & -\mathbf{C}^T(\bar{q}_I) & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{X}_8(\bar{q}_I, \mathbf{b}_g)\end{bmatrix} \tag{40}$$

In the ensuing derivations, $\mathbf{X}_i(\cdot),\ i = 1, 2, \ldots$ are eliminated, hence we do not need to compute them explicitly. If we stack the computed gradients we will have:

$$\mathcal{O} = \begin{bmatrix} \nabla\mathfrak{L}_0\mathbf{h}_1 \\ \nabla\mathfrak{L}_0\mathbf{h}_2 \\ \nabla\mathfrak{L}_{f_0}\mathbf{h}_1 \\ \nabla\mathfrak{L}_{f_0}\mathbf{h}_2 \\ \nabla\mathfrak{L}_{\mathbf{f}_1}\mathbf{h}_2 \\ \nabla\mathfrak{L}_0 h_3 \\ \nabla\mathfrak{L}_{f_0}\mathfrak{L}_{f_0}\mathbf{h}_2 \end{bmatrix} = \begin{bmatrix} \mathcal{R}(\bar{q}_C)\mathbf{J} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathcal{L}(\mathbf{J}\bar{q}_I) & \mathbf{0}_{4\times3} \\ \mathbf{\Psi}(\bar{q}_I, \mathbf{p}_C) & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{I}_{3\times3} & \mathbf{0}_{3\times4} & \mathbf{C}^T(\bar{q}_I) \\ \mathbf{X}_1 & -\frac{1}{2}\mathcal{R}(\bar{q}_C)\mathbf{J}\mathbf{\Xi}(\bar{q}_I) & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{X}_2 & \mathbf{0}_{4\times3} \\ \mathbf{X}_3 & \mathbf{X}_4 & \mathbf{I}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times4} & \mathbf{X}_5 \\ \mathbf{\Gamma}(\bar{q}_I, \mathbf{p}_C) & \mathbf{0}_{9\times3} & \mathbf{0}_{9\times3} & \mathbf{0}_{9\times3} & \mathbf{0}_{9\times3} & \mathbf{0}_{9\times4} & \mathbf{\Upsilon}(\bar{q}_I) \\ 2\bar{q}_I^T & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times4} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} \\ \mathbf{X}_6 & \mathbf{X}_7 & \mathbf{0}_{3\times3} & -\mathbf{C}^T(\bar{q}_I) & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times4} & \mathbf{X}_8 \end{bmatrix} \tag{41}$$

In order to show that the observability rank condition [3] is met, we just need to prove that matrix $\mathcal{O}$ is full rank. To prove the full rank property of $\mathcal{O}$, we start by showing that $\begin{bmatrix}\nabla\mathfrak{L}_{\mathbf{f}_1}\mathbf{h}_2^T & \nabla\mathfrak{L}_0 h_3^T\end{bmatrix}^T$ is full rank and then eliminating all other matrices in the first and last columns of $\mathcal{O}$. Then we prove that $\mathcal{L}(\mathbf{J}\bar{q}_I)$ is full rank and eliminate $\mathbf{X}_2$. Finally, after showing that $-\frac{1}{2}\mathcal{R}(\bar{q}_C)\mathbf{J}\mathbf{\Xi}(\bar{q}_I)$ is full rank, we eliminate $\mathbf{X}_4$ and $\mathbf{X}_7$, and the proof is complete. The remainder of this section specifically details these steps.

Step 1 Based on Lemma 1, $\begin{bmatrix}\nabla\mathfrak{L}_{\mathbf{f}_1}\mathbf{h}_2\\ \nabla\mathfrak{L}_0 h_3\end{bmatrix}$ is full rank. Therefore by performing Gaussian elimination on the corresponding rows of $\mathcal{O}$, we get

$$\begin{bmatrix}
\mathcal{R}(\bar{q}_C)\mathbf{J} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathcal{L}(\mathbf{J}\bar{q}_I) & \mathbf{0}_{4\times3}\\
\mathbf{\Psi}(\bar{q}_I,\mathbf{p}_C) & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{I}_{3\times3} & \mathbf{0}_{3\times4} & \mathbf{C}^T(\bar{q}_I)\\
\mathbf{X}_1 & -\frac{1}{2}\mathcal{R}(\bar{q}_C)\mathbf{J}\mathbf{\Xi}(\bar{q}_I) & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{X}_2 & \mathbf{0}_{4\times3}\\
\mathbf{X}_3 & \mathbf{X}_4 & \mathbf{I}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times4} & \mathbf{X}_5\\
\mathbf{I}_{4\times4} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times4} & \mathbf{0}_{4\times3}\\
\mathbf{0}_{3\times4} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times4} & \mathbf{I}_{3\times3}\\
\mathbf{0}_{3\times4} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times4} & \mathbf{0}_{3\times3}\\
\mathbf{X}_6 & \mathbf{X}_7 & \mathbf{0}_{3\times3} & -\mathbf{C}^T(\bar{q}_I) & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times4} & \mathbf{X}_8
\end{bmatrix} \tag{42}$$

Then we can easily eliminate all other elements of the first and last columns:

$$\begin{bmatrix}
\mathbf{0}_{4\times4} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathcal{L}(\mathbf{J}\bar{q}_I) & \mathbf{0}_{4\times3}\\
\mathbf{0}_{3\times4} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{I}_{3\times3} & \mathbf{0}_{3\times4} & \mathbf{0}_{3\times3}\\
\mathbf{0}_{4\times4} & -\frac{1}{2}\mathcal{R}(\bar{q}_C)\mathbf{J}\mathbf{\Xi}(\bar{q}_I) & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{X}_2 & \mathbf{0}_{4\times3}\\
\mathbf{0}_{3\times4} & \mathbf{X}_4 & \mathbf{I}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times4} & \mathbf{0}_{3\times3}\\
\mathbf{I}_{4\times4} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times4} & \mathbf{0}_{4\times3}\\
\mathbf{0}_{3\times4} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times4} & \mathbf{I}_{3\times3}\\
\mathbf{0}_{3\times4} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times4} & \mathbf{0}_{3\times3}\\
\mathbf{0}_{3\times4} & \mathbf{X}_7 & \mathbf{0}_{3\times3} & -\mathbf{C}^T(\bar{q}_I) & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times4} & \mathbf{0}_{3\times3}
\end{bmatrix} \tag{43}$$

Step 2 Based on Lemma 2, $L(\mathbf{J}\bar{q}_I)$ is full rank, thus we diagonalize it with Gaussian elimination:

$$\begin{bmatrix}
\mathbf{0}_{4\times4} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{I}_{4\times4} & \mathbf{0}_{4\times3}\\
\mathbf{0}_{3\times4} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{I}_{3\times3} & \mathbf{0}_{3\times4} & \mathbf{0}_{3\times3}\\
\mathbf{0}_{4\times4} & -\frac{1}{2}\mathcal{R}(\bar{q}_C)\mathbf{J}\mathbf{\Xi}(\bar{q}_I) & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{X}_2 & \mathbf{0}_{4\times3}\\
\mathbf{0}_{3\times4} & \mathbf{X}_4 & \mathbf{I}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times4} & \mathbf{0}_{3\times3}\\
\mathbf{I}_{4\times4} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times4} & \mathbf{0}_{4\times3}\\
\mathbf{0}_{3\times4} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times4} & \mathbf{I}_{3\times3}\\
\mathbf{0}_{3\times4} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times4} & \mathbf{0}_{3\times3}\\
\mathbf{0}_{3\times4} & \mathbf{X}_7 & \mathbf{0}_{3\times3} & -\mathbf{C}^T(\bar{q}_I) & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times4} & \mathbf{0}_{3\times3}
\end{bmatrix} \tag{44}$$

And now we can eliminate $\mathbf{X}_2$:

$$\begin{bmatrix}
\mathbf{0}_{4\times4} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{I}_{4\times4} & \mathbf{0}_{4\times3}\\
\mathbf{0}_{3\times4} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{I}_{3\times3} & \mathbf{0}_{3\times4} & \mathbf{0}_{3\times3}\\
\mathbf{0}_{4\times4} & -\frac{1}{2}\mathcal{R}(\bar{q}_C)\mathbf{J}\mathbf{\Xi}(\bar{q}_I) & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times4} & \mathbf{0}_{4\times3}\\
\mathbf{0}_{3\times4} & \mathbf{X}_4 & \mathbf{I}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times4} & \mathbf{0}_{3\times3}\\
\mathbf{I}_{4\times4} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times4} & \mathbf{0}_{4\times3}\\
\mathbf{0}_{3\times4} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times4} & \mathbf{I}_{3\times3}\\
\mathbf{0}_{3\times4} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times4} & \mathbf{0}_{3\times3}\\
\mathbf{0}_{3\times4} & \mathbf{X}_7 & \mathbf{0}_{3\times3} & -\mathbf{C}^T(\bar{q}_I) & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times4} & \mathbf{0}_{3\times3}
\end{bmatrix} \tag{45}$$

Step 3 Finally, using the fact that $\mathcal{R}(\bar{q}_C)\mathbf{J}\mathbf{\Xi}(\bar{q}_I)$ is full rank (cf. Lemma 3), we reduce it to identity by employing Gaussian elimination.

$$\begin{bmatrix}
\mathbf{0}_{4\times4} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{I}_{4\times4} & \mathbf{0}_{4\times3}\\
\mathbf{0}_{3\times4} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{I}_{3\times3} & \mathbf{0}_{3\times4} & \mathbf{0}_{3\times3}\\
\mathbf{0}_{3\times4} & \mathbf{I}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times4} & \mathbf{0}_{3\times3}\\
\mathbf{0}_{1\times4} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times4} & \mathbf{0}_{1\times3}\\
\mathbf{0}_{3\times4} & \mathbf{X}_4 & \mathbf{I}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times4} & \mathbf{0}_{3\times3}\\
\mathbf{I}_{4\times4} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times4} & \mathbf{0}_{4\times3}\\
\mathbf{0}_{3\times4} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times4} & \mathbf{I}_{3\times3}\\
\mathbf{0}_{3\times4} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times4} & \mathbf{0}_{3\times3}\\
\mathbf{0}_{3\times4} & \mathbf{X}_7 & \mathbf{0}_{3\times3} & -\mathbf{C}^T(\bar{q}_I) & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times4} & \mathbf{0}_{3\times3}
\end{bmatrix} \tag{46}$$

And similar to previous steps, we eliminate $\mathbf{X}_4$ and $\mathbf{X}_7$:

$$\begin{bmatrix} \mathbf{0}_{4\times4} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{I}_{4\times4} & \mathbf{0}_{4\times3} \\ \mathbf{0}_{3\times4} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{I}_{3\times3} & \mathbf{0}_{3\times4} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times4} & \mathbf{I}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times4} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{1\times4} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times4} & \mathbf{0}_{1\times3} \\ \mathbf{0}_{3\times4} & \mathbf{0}_{3\times3} & \mathbf{I}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times4} & \mathbf{0}_{3\times3} \\ \mathbf{I}_{4\times4} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times4} & \mathbf{0}_{4\times3} \\ \mathbf{0}_{3\times4} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times4} & \mathbf{I}_{3\times3} \\ \mathbf{0}_{3\times4} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times4} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times4} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & -\mathbf{C}^T(\bar{q}_I) & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times4} & \mathbf{0}_{3\times3} \end{bmatrix} \tag{47}$$

considering that rotation matrix $\mathbf{C}^T(\bar{q}_I)$ is always full rank, it is easy to see that (47) is full rank which indicates that $\mathcal{O}$ is full rank too.

Lemma 1. $\begin{bmatrix} \nabla\mathfrak{L}_{\mathbf{f}_1}\mathbf{h}_2 \\ \nabla\mathfrak{L}_0 h_3 \end{bmatrix}$ *is full rank.*

Proof. First we compute $\mathbf{\Gamma}$ and $\mathbf{\Upsilon}$ using a MATLAB script (cf. Appendix D and C respectively). The final results are:

$$\mathbf{\Gamma} \triangleq \begin{bmatrix} \mathbf{\Gamma}_1 \\ \mathbf{\Gamma}_2 \\ \mathbf{\Gamma}_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\frac{\partial \mathbf{\Psi}(\bar{q}_I,\mathbf{p}_C)\mathbf{\Xi}(\bar{q}_I)\mathbf{e}_1}{\partial \bar{q}_I} \\ -\frac{1}{2}\frac{\partial \mathbf{\Psi}(\bar{q}_I,\mathbf{p}_C)\mathbf{\Xi}(\bar{q}_I)\mathbf{e}_2}{\partial \bar{q}_I} \\ -\frac{1}{2}\frac{\partial \mathbf{\Psi}(\bar{q}_I,\mathbf{p}_C)\mathbf{\Xi}(\bar{q}_I)\mathbf{e}_3}{\partial \bar{q}_I} \end{bmatrix}$$

$$= \left[\begin{array}{cccc} -2p_2q_3+2p_3q_2 & -2q_4p_2+2p_3q_1 & -2q_4p_3-2p_2q_1 & -2p_3q_3-2p_2q_2 \\ 2q_4p_2-2p_3q_1 & -2p_2q_3+2p_3q_2 & -2p_3q_3-2p_2q_2 & 2q_4p_3+2p_2q_1 \\ 2q_4p_3+2p_2q_1 & 2p_3q_3+2p_2q_2 & -2p_2q_3+2p_3q_2 & -2q_4p_2+2p_3q_1 \\ 2p_1q_3-2p_3q_1 & 2q_4p_1+2p_3q_2 & 2p_3q_3+2p_1q_1 & -2q_4p_3+2p_1q_2 \\ -2q_4p_1-2p_3q_2 & 2p_1q_3-2p_3q_1 & -2q_4p_3+2p_1q_2 & -2p_3q_3-2p_1q_1 \\ -2p_3q_3-2p_1q_1 & 2q_4p_3-2p_1q_2 & 2p_1q_3-2p_3q_1 & 2q_4p_1+2p_3q_2 \\ -2p_1q_2+2p_2q_1 & -2p_2q_2-2p_1q_1 & 2q_4p_1-2p_2q_3 & 2q_4p_2+2p_1q_3 \\ 2p_2q_2+2p_1q_1 & -2p_1q_2+2p_2q_1 & 2q_4p_2+2p_1q_3 & -2q_4p_1+2p_2q_3 \\ -2q_4p_1+2p_2q_3 & -2q_4p_2-2p_1q_3 & -2p_1q_2+2p_2q_1 & -2p_2q_2-2p_1q_1 \end{array}\right] \begin{array}{l} \left.\begin{array}{c}1\\2\\3\end{array}\right\}\omega_x \\ \left.\begin{array}{c}4\\5\\6\end{array}\right\}\omega_y \\ \left.\begin{array}{c}7\\8\\9\end{array}\right\}\omega_z \end{array} \tag{48}$$

$$\mathbf{\Upsilon} \triangleq \begin{bmatrix} \mathbf{\Upsilon}_1 \\ \mathbf{\Upsilon}_2 \\ \mathbf{\Upsilon}_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\frac{\partial \mathbf{\Psi}(\bar{q}_I,\mathbf{p}_C)\mathbf{\Xi}(\bar{q}_I)\mathbf{e}_1}{\partial \mathbf{p}_C} \\ -\frac{1}{2}\frac{\partial \mathbf{\Psi}(\bar{q}_I,\mathbf{p}_C)\mathbf{\Xi}(\bar{q}_I)\mathbf{e}_2}{\partial \mathbf{p}_C} \\ -\frac{1}{2}\frac{\partial \mathbf{\Psi}(\bar{q}_I,\mathbf{p}_C)\mathbf{\Xi}(\bar{q}_I)\mathbf{e}_3}{\partial \mathbf{p}_C} \end{bmatrix}$$

$$= \left[\begin{array}{ccc} 0 & -2q_1q_3-2q_4q_2 & 2q_1q_2-2q_4q_3 \\ 0 & -2q_3q_2+2q_1q_4 & q_4^2-q_3^2+q_2^2-q_1^2 \\ 0 & -q_4^2-q_3^2+q_1^2+q_2^2 & 2q_3q_2+2q_1q_4 \\ 2q_1q_3+2q_4q_2 & 0 & -q_4^2+q_3^2-q_1^2+q_2^2 \\ 2q_3q_2-2q_1q_4 & 0 & -2q_1q_2-2q_4q_3 \\ q_4^2+q_3^2-q_1^2-q_2^2 & 0 & 2q_4q_2-2q_1q_3 \\ -2q_1q_2+2q_4q_3 & q_4^2-q_3^2+q_1^2-q_2^2 & 0 \\ -q_2^2+q_1^2-q_4^2+q_3^2 & 2q_1q_2+2q_4q_3 & 0 \\ -2q_3q_2-2q_1q_4 & -2q_4q_2+2q_1q_3 & 0 \end{array}\right] \begin{array}{l} \left.\begin{array}{c}1\\2\\3\end{array}\right\}\omega_x \\ \left.\begin{array}{c}4\\5\\6\end{array}\right\}\omega_y \\ \left.\begin{array}{c}7\\8\\9\end{array}\right\}\omega_z \end{array} \tag{49}$$

where $\bar{q}_I = [q_1\ ,\ q_2\ ,\ q_3\ ,\ q_4]$ and $\mathbf{p}_C = [p_1\ ,\ p_2\ ,\ p_3]$. The variables shown right next to the the row numbers (i.e., ω_x, ω_y, and ω_z), indicate what control input should be excited in order to use the corresponding row numbers. In other words, in the subsequent derivations, whenever we use a row (e.g., the 4th row), the underlaying assumption is that the corresponding control input (e.g., $\omega_y \neq 0$) is nonzero.

We stack $\nabla\mathfrak{L}_0 h_3$ beneath $\nabla\mathfrak{L}_{\mathbf{f}_1}\mathbf{h}_2$ and define the following 10×7 matrix:

$$\mathbf{A} = \begin{bmatrix} \mathbf{\Gamma} & \mathbf{\Upsilon} \\ 2\bar{q}_I^T & \mathbf{0}_{1\times 3} \end{bmatrix} \begin{matrix} \}\,(1 \quad - \quad 9) \\ \}\,10 \end{matrix} \tag{50}$$

In order to prove this lemma it is sufficient to find a combination of the rows of this matrix that are always independent, and hence, the square submatrix consisting of them has nonzero determinant. We need to assume that $\mathbf{p}_C$ has nonzero length[1]. This is a reasonable assumption as it is physically impossible that IMU center coincides with the camera focal point. We hereafter examine the following 3 cases of interest:

1. If at no component of $\mathbf{p}_C$ is zero: in that case we can use any of the following selections of the rows:

$$\mathtt{det}\left(\mathfrak{W}\{1,2,3,4,5,6,10\}\right) = \alpha\, p_3\left(p_2^2+p_1^2\right)\left(q_1^2+q_4^2+q_2^2+q_3^2\right)^5 = \alpha\, p_3\left(p_2^2+p_1^2\right) \tag{51}$$
$$\mathtt{det}\left(\mathbf{A}\{1,2,3,7,8,9,10\}\right) = \alpha\, p_2\left(p_3^2+p_1^2\right)\left(q_1^2+q_4^2+q_2^2+q_3^2\right)^5 = \alpha\, p_2\left(p_3^2+p_1^2\right) \tag{52}$$
$$\mathtt{det}\left(\mathbf{A}\{4,5,6,7,8,9,10\}\right) = \alpha\, p_1\left(p_2^2+p_3^2\right)\left(q_1^2+q_4^2+q_2^2+q_3^2\right)^5 = \alpha\, p_1\left(p_2^2+p_3^2\right) \tag{53}$$

   It is worth-mentioning that each selection of the rows is based on 2 nonzero components of $\omega = [\omega_x\ ,\ \omega_y\ ,\ \omega_z]$. Thus our freedom to choose any of these three selections indicates that excitation of any two direction of the control input ω is sufficient to make this submatrix full rank as long as all components of $\mathbf{p}_C$ are nonzero.

2. If exactly one component of $\mathbf{p}_C$ is zero, we can still use two of three equations in (51)-(53). For example if $p_1 = 0$, then we can use either (51) which assumes that ω_x and ω_y are nonzero, or (52) that assumes ω_x and ω_z are nonzero. However, (53) is zero in this cases which means that excitation of ω_y and ω_z is not sufficient to render the system observable[2].

3. When two components of $\mathbf{p}_C$ are zero, none of (51)-(53) can be used. However this case can be dealt with using other selections of the rows. Without loss of generality, let us assume that the nonzero component of $\mathbf{p}_C$ is p_3 (i.e., $p_1 = 0$, $p_2 = 0$). Then we can consider the following selections of the rows:

$$\mathtt{det}\left(\mathbf{A}\{1,2,3,6,7,8,10\}\right) = \alpha\, p_3^3(-q_4^2-q_3^2+q_2^2+q_1^2)^2(q_1^2+q_2^2+q_4^2+q_3^2)^3 \tag{54}$$
$$\mathtt{det}\left(\mathbf{A}\{1,2,3,5,7,9,10\}\right) = \alpha\, p_3^3(q_1q_4-q_3q_2)^2(q_1^2+q_2^2+q_3^2+q_4^2)^3 \tag{55}$$
$$\mathtt{det}\left(\mathbf{A}\{1,2,3,4,8,9,10\}\right) = \alpha\, p_3^3(q_2q_4+q_1q_3)^2(q_1^2+q_3^2+q_2^2+q_4^2)^3 \tag{56}$$

   After using the unit quaternion constraint, these expression are simplified to:

$$\mathtt{det}\left(\mathbf{A}\{1,2,3,6,7,8,10\}\right) = \alpha\, p_3^3(-q_4^2-q_3^2+q_2^2+q_1^2)^2 \tag{57}$$
$$\mathtt{det}\left(\mathbf{A}\{1,2,3,5,7,9,10\}\right) = \alpha\, p_3^3(q_1q_4-q_3q_2)^2 \tag{58}$$
$$\mathtt{det}\left(\mathbf{A}\{1,2,3,4,8,9,10\}\right) = \alpha\, p_3^3(q_2q_4+q_1q_3)^2 \tag{59}$$

   In order to show that the determinant of these three selections of the rows can not be zero simultaneously, we need to consider 3 cases:

   (a) For the case that all components of $\bar{q}_I$ are nonzero, at least one of (58) and (59) will be nonzero. To prove this, we assume that both of them are zero when all components of $\bar{q}_I$ are nonzero and reach to a contradiction. If both (58) and (59) are zero, then we can substitute $q_1 = \frac{q_3q_2}{q_4}$ from (58) in (59) and multiply it by q_4, then we will get:

$$q_2(q_4^2+q_3^2) = 0 \tag{60}$$

   which is impossible because we have assumed that all the components of $\bar{q}_I$ are nonzero. So $\mathbf{A}$ is full rank if $\bar{q}_I$ has no zero element.

[1] Otherwise $\mathbf{\Gamma}$ will be zero and $\mathbf{A}$ will be rank deficient.

[2] Although there are $\binom{10}{7} = 120$ different combination of the rows, and these are just three of them, the conclusions drawn from all them is the same.

(b) When exactly one component of $\bar{q}_I$ is zero, both (58) and (59) are nonzero. For example if $q_1 = 0$,

$$\texttt{det}\,(\mathbf{A}\{1,2,3,5,7,9,10\}) = \alpha\, p_3^3 (q_3 q_2)^2 \neq 0 \quad (61)$$
$$\texttt{det}\,(\mathbf{A}\{1,2,3,4,8,9,10\}) = \alpha\, p_3^3 (q_2 q_4)^2 \neq 0 \quad (62)$$

(c) If two components of $\bar{q}_I$ are nonzero, at least one of (57)-(59) is nonzero. For example, if $q_1 = 0$ and $q_2 = 0$ and the two others are nonzero, then we can use (57):

$$\texttt{det}\,(\mathbf{A}\{1,2,3,6,7,8,10\}) = \alpha\, p_3^3 (-q_4^2 - q_3^2 + q_2^2 + q_1^2)^2 = \alpha\, p_3^3 (q_4^2 + q_3^2)^2 = \alpha\, p_3^3 \neq 0 \quad (63)$$

or if $q_1 = 0$ and $q_3 = 0$, we can use (59):

$$\texttt{det}\,(\mathbf{A}\{1,2,3,4,8,9,10\}) = \alpha\, p_3^3 (q_2 q_4)^2 \neq 0 \quad (64)$$

(d) Finally, if three components of $\bar{q}_I$ are zero, (57) is nonzero due to unit quaternion constraint. For example if the only nonzero element is q_1,

$$\texttt{det}\,(\mathbf{A}\{1,2,3,6,7,8,10\}) = \alpha\, p_3^3 (q_1^2)^2 = \alpha\, p_3^3 \neq 0 \quad (65)$$

This part of the proof was particularly established for the case of $\mathbf{p}_C = [0\ ,\ 0\ ,\ p_3]^T, p_3 \neq 0$. However, the proofs for two other cases, i.e., $\mathbf{p}_C = [p_1\ ,\ 0\ ,\ 0]^T, p_1 \neq 0$ and $\mathbf{p}_C = [0\ ,\ p_2\ ,\ 0]^T, p_2 \neq 0$, can be followed similarly and are not repeated here.

Thus we proved that matrix $\mathbf{A}$ is full rank for all possible values of $\mathbf{p}_C$. It is worth-noting that except when all components of $\mathbf{p}_C$ are nonzero, we need to excite all three directions of $\boldsymbol{\omega} = [\omega_x\ ,\ \omega_y\ ,\ \omega_z]$ to have full rank $\mathbf{A}$. The MATLAB script used in symbolic computation of the determinants in (57)-(59) is provided in Appendix D. □

Lemma 2. *For any unit quaternion $\bar{q}$, $\mathcal{L}(\mathbf{J}\bar{q})$ is full rank.*

Proof. Since $\mathbf{J}\bar{q}$ is a unit quaternion itself, we prove that $\mathcal{L}(\bar{q})$ is full rank.

$$\texttt{det}\,(\mathcal{L}(\bar{q})) = \left(q_1^2 + q_4^2 + q_3^2 + q_1^2\right)^2 = ||\bar{q}||^4 = 1 \quad (66)$$

where $\bar{q} = [q_1\ ,\ q_2\ ,\ q_3\ ,\ q_4]$. Thus the proof is complete. □

Lemma 3. *For any unit quaternions $\bar{q}$ and $\bar{s}$, $\mathcal{R}(\bar{q})\mathbf{J}\boldsymbol{\Xi}(\bar{s})$ is full rank.*

Proof. First we expand $\mathcal{R}(\bar{q})\mathbf{J}\boldsymbol{\Xi}(\bar{s})$:

$$\mathbf{B} = \mathcal{R}(\bar{q})\mathbf{J}\boldsymbol{\Xi}(\bar{s}) = \begin{bmatrix} -q_4 s_4 + q_3 s_3 + q_2 s_2 - q_1 s_1 & q_4 s_3 + q_3 s_4 - q_2 s_1 - q_1 s_2 & -q_4 s_2 - q_3 s_1 - q_2 s_4 - q_1 s_3 \\ -q_3 s_4 - q_4 s_3 - q_1 s_2 - q_2 s_1 & q_3 s_3 - q_4 s_4 + q_1 s_1 - q_2 s_2 & -q_3 s_2 + q_4 s_1 + q_1 s_4 - q_2 s_3 \\ q_2 s_4 - q_1 s_3 + q_4 s_2 - q_3 s_1 & -q_2 s_3 - q_1 s_4 - q_4 s_1 - q_3 s_2 & q_2 s_2 + q_1 s_1 - q_4 s_4 - q_3 s_3 \\ q_1 s_4 + q_2 s_3 - q_3 s_2 - q_4 s_1 & -q_1 s_3 + q_2 s_4 + q_3 s_1 - q_4 s_2 & q_1 s_2 - q_2 s_1 + q_3 s_4 - q_4 s_3 \end{bmatrix} \quad (67)$$

where $\bar{q} = [q_1\ ,\ q_2\ ,\ q_3\ ,\ q_4]$ and $\bar{s} = [s_1\ ,\ s_2\ ,\ s_3\ ,\ s_4]$. Four combinations of the rows of $\mathbf{B}$ exist, and their determinants are:

$$\texttt{det}\,(\mathbf{B}\{1,2,3\}) = -\left(s_1^2 + s_4^2 + s_2^2 + s_3^2\right)\left(q_3^2 + q_4^2 + q_2^2 + q_1^2\right)(q_1 s_1 + q_3 s_3 + q_2 s_2 + q_4 s_4) \quad (68)$$
$$\texttt{det}\,(\mathbf{B}\{1,2,4\}) = -\left(s_1^2 + s_4^2 + s_2^2 + s_3^2\right)\left(q_3^2 + q_4^2 + q_2^2 + q_1^2\right)(q_1 s_2 - q_2 s_1 + q_4 s_3 - q_3 s_4) \quad (69)$$
$$\texttt{det}\,(\mathbf{B}\{1,3,4\}) = -\left(s_1^2 + s_4^2 + s_2^2 + s_3^2\right)\left(q_3^2 + q_4^2 + q_2^2 + q_1^2\right)(q_1 s_3 + q_2 s_4 - q_3 s_1 - q_4 s_2) \quad (70)$$
$$\texttt{det}\,(\mathbf{B}\{2,3,4\}) = \left(s_1^2 + s_4^2 + s_2^2 + s_3^2\right)\left(q_3^2 + q_4^2 + q_2^2 + q_1^2\right)(q_1 s_4 + q_3 s_2 - q_2 s_3 - q_4 s_1) \quad (71)$$

We used the fact that $\bar{q}$ and $\bar{s}$ are unit quaternions to simplify these expressions:

$$\texttt{det}\,(\mathbf{B}\{1,2,3\}) = (q_1 s_1 + q_3 s_3 + q_2 s_2 + q_4 s_4) \quad (72)$$
$$\texttt{det}\,(\mathbf{B}\{1,2,4\}) = (q_1 s_2 - q_2 s_1 + q_4 s_3 - q_3 s_4) \quad (73)$$

$$\text{det}\,(\mathbf{B}\{1,3,4\}) = (q_1 s_3 + q_2 s_4 - q_3 s_1 - q_4 s_2) \tag{74}$$

$$\text{det}\,(\mathbf{B}\{2,3,4\}) = (q_1 s_4 + q_3 s_2 - q_2 s_3 - q_4 s_1) \tag{75}$$

It is clear that these four determinant become zero simultaneously only if the following system of equations has a solution:

$$\begin{matrix} q_1 s_1 + q_3 s_3 + q_2 s_2 + q_4 s_4 = 0 \\ q_1 s_2 - q_2 s_1 + q_4 s_3 - q_3 s_4 = 0 \\ q_1 s_3 + q_2 s_4 - q_3 s_1 - q_4 s_2 = 0 \\ q_1 s_4 + q_3 s_2 - q_2 s_3 - q_4 s_1 = 0 \end{matrix} \quad \Longleftrightarrow \quad \underbrace{\begin{bmatrix} q_1 & q_2 & q_3 & q_4 \\ -q_2 & q_1 & q_4 & -q_3 \\ -q_3 & -q_4 & q_1 & q_2 \\ -q_4 & q_3 & -q_2 & q_1 \end{bmatrix}}_{\mathbf{D}} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} = 0 \tag{76}$$

However,

$$\text{det}\,(\mathbf{D}) = \left(q_3^2 + q_4^2 + q_2^2 + q_1^2\right)^2 = 1 \tag{77}$$

which means the only solution to (76) is $\bar{s} = [0\ ,\ 0\ ,\ 0\ ,\ 0]$. However, this solution is not possible since $\bar{s}$ is a unit quaternion. Thus the determinant (72)-(75) can not be simultaneously zero, that is, $\mathbf{B}$ is always full rank. A MATLAB script for symbolic computation of the determinants is provided in Appendix B. □

Lemma 4. $\begin{bmatrix} 2\bar{q}_I^T \\ \mathbf{\Theta}(\bar{q}_I) \end{bmatrix}$ *is full rank.*

Proof. In order to show that this matrix is full rank it is sufficient to show that there always exist a 4×4 sub-matrix of this matrix which has a non-zero determinant. An expanded version of $\left[2\bar{q}_I \quad \mathbf{\Theta}(\bar{q}_I)^T\right]^T$ is shown below. The first nine rows of this matrix are associated with three directions of the measured linear acceleration, $\mathbf{a}_m = [a_1\ ,\ a_2\ ,\ a_3]^T$. This means that a_1, a_2, and a_3 must be non-zero to have nonzero rows respectively at first, second, and third blocks of (78). In other words, if we use row 1 in proving the (78) is full rank, the underlaying assumption is that a_1 is excited and is not always zero as a control input.

$$\mathfrak{X} = \begin{bmatrix} \mathbf{\Theta}(\bar{q}) \\ 2\bar{q}^T \end{bmatrix} = \left[\begin{matrix} 2q_3 & 2q_4 & 2q_1 & 2q_2 \\ -2q_4 & 2q_3 & 2q_2 & -2q_1 \\ -4q_1 & -4q_2 & 0 & 0 \\ 2q_2 & 2q_1 & -2q_4 & -2q_3 \\ -4q_1 & 0 & -4q_3 & 0 \\ 2q_4 & 2q_3 & 2q_2 & 2q_1 \\ 0 & -4q_2 & -4q_3 & 0 \\ 2q_2 & 2q_1 & 2q_4 & 2q_3 \\ 2q_3 & -2q_4 & 2q_1 & -2q_2 \\ 2q_1 & 2q_2 & 2q_3 & 2q_4 \end{matrix}\right] \begin{matrix} \left.\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}\right\} a_1 \\ \left.\begin{matrix} 4 \\ 5 \\ 6 \end{matrix}\right\} a_2 \\ \left.\begin{matrix} 7 \\ 8 \\ 9 \end{matrix}\right\} a_3 \\ 10\}\ \bar{q}^T \end{matrix} \tag{78}$$

We employ MATLAB® symbolic functions `det` and `factor` which are based on Maple® routines to compute and simplify the determinants. The complete MATLAB® script that was used in this lemma can be found in Appendix A.

We need at least two nonzero directions of $\mathbf{a}_m$ to show that (78) is full rank. This is because the only combinations which need just one direction of $\mathbf{a}_m$ are the selection of the rows $\{1,2,3,10\}$, $\{4,5,6,10\}$, and $\{7,8,9,10\}$. Unfortunately all these tree combinations result in zero determinant. However, if we assume that $\mathbf{a}_m$ has at least two nonzero elements, we can easily show that (78) is indeed full rank. For this purpose, without loss of generality, let us assume that $a_1, a_2 \neq 0$. The process for other combinations of two non-zero elements of $\mathbf{a}_m$ is similar.

We start by computing the determinant corresponding to the rows $\{1,2,3,6\}$:

$$\text{det}\,(\mathfrak{X}\{1,2,3,6\}) = \alpha \left(q_2^2 + q_3^2\right) \left(q_2^2 + q_1^2 - q_4^2 - q_3^2\right) \tag{79}$$

where α is a constant number and $\mathfrak{X}\{1,2,3,6\}$ represents a matrix consisting of 1st, 2nd, 3rd, and 6th rows of $\mathfrak{X}$. If (79) is nonzero, then $\mathfrak{X}$ is full rank and the lemma is proved. So we just need to investigate the cases where (79) becomes zero. For having polynomial (79) to be zero, at least one of the following conditions must be met:

$$\begin{cases} q_2 = 0 \\ q_3 = 0 \end{cases} \qquad \text{or} \qquad q_2^2 + q_1^2 = q_4^2 + q_3^2 \tag{80}$$

If he first case is true and both q_2 and q_3 are zero, then we can use rows $\{2, 3, 5, 6\}$:

$$\texttt{det}\left(\mathfrak{X}\{2,3,5,6\}\right)\Big|_{q_2=0\ ,\ q_3=0} = \alpha\, q_1^2 \left(q_4^2 + q_1^2\right) \tag{81}$$

The only case that this expression becomes zero too is when $q_1 = 0$. Note that in this case q_4 can not be zero because we are using unit quaternions. However, when q_1 is zero in addition to q_2 and q_3, we can use rows $\{2, 3, 6, 10\}$:

$$\texttt{det}\left(\mathfrak{X}\{2,3,6,10\}\right)\Big|_{q_1=0\ ,\ q_2=0\ ,\ q_3=0} = \alpha\, q_4^4 \tag{82}$$

which can not be zero. Thus we have proved that if both q_2 and q_3 are zero, we still have full rank $\mathfrak{X}$.

Now we can investigate the second case when at least one of q_2 and q_3 is nonzero and $q_2^2 + q_1^2 = q_4^2 + q_3^2$. If $q_2 \neq 0$ and $q_3 = 0$, then q_4 must be nonzero. In this case we can use rows $\{1, 2, 3, 4\}$:

$$\texttt{det}\left(\mathfrak{X}\{1,2,3,4\}\right)\Big|_{q_3=0} = \alpha \left(q_2^2 + q_3^2\right)\left(q_4 q_2 + q_1 q_3\right)\Big|_{q_3=0} = \alpha\, q_2^3 q_4 \tag{83}$$

which can not be zero. Similarly, if $q_3 \neq 0$ and $q_2 = 0$, then q_1 must be nonzero. In this case we can still use rows $\{1, 2, 3, 4\}$:

$$\texttt{det}\left(\mathfrak{X}\{1,2,3,4\}\right)\Big|_{q_2=0} = \alpha \left(q_2^2 + q_3^2\right)\left(q_4 q_2 + q_1 q_3\right)\Big|_{q_2=0} = \alpha\, q_3^3 q_1 \tag{84}$$

The last case that remains to be investigated is when both q_2 and q_3 are nonzero. In this case we can show that selections of $\{1, 2, 3, 4\}$ and $\{1, 2, 3, 5\}$ can not be zero simultaneously:

$$\texttt{det}\left(\mathfrak{X}\{1,2,3,4\}\right) = \alpha \left(q_2^2 + q_3^2\right)\left(q_4 q_2 + q_1 q_3\right) \tag{85}$$

$$\texttt{det}\left(\mathfrak{X}\{1,2,3,5\}\right) = \alpha \left(q_2^2 + q_3^2\right)\left(q_4 q_1 - q_2 q_3\right) \tag{86}$$

Knowing that $q_2 \neq 0$ and $q_3 \neq 0$, to have both (85) and (86) zero, we should have:

$$q_4 q_2 + q_1 q_3 = 0 \tag{87}$$

$$q_4 q_1 - q_2 q_3 = 0 \tag{88}$$

From the first expression we find $q_4 = -\frac{q_1 q_3}{q_2}$. If we substitute this into the second equation and multiply by q_2, we will get $q_1^2 q_3 + q_2^2 q_3 = 0$ which is impossible because q_2 and q_3 are assumed to be nonzero. Therefore given $q_2 \neq 0$ and $q_3 \neq 0$, (85) and (86) can not be simultaneously zero and the lemma is proved. □

Lemma 5. *$\boldsymbol{\Upsilon}(\bar{q})$ is full rank.*

Proof. We again expand this matrix as follows:

$$\mathfrak{Z} = \boldsymbol{\Upsilon}(\bar{q}) = \begin{bmatrix} 0 & -2q_1q_3 - 2q_4q_2 & 2q_1q_2 - 2q_4q_3 \\ 0 & -2q_3q_2 + 2q_1q_4 & q_4^2 - q_3^2 + q_2^2 - q_1^2 \\ 0 & -q_4^2 - q_3^2 + q_1^2 + q_2^2 & 2q_3q_2 + 2q_1q_4 \\ 2q_1q_3 + 2q_4q_2 & 0 & -q_4^2 + q_3^2 - q_1^2 + q_2^2 \\ 2q_3q_2 - 2q_1q_4 & 0 & -2q_1q_2 - 2q_4q_3 \\ q_4^2 + q_3^2 - q_1^2 - q_2^2 & 0 & 2q_4q_2 - 2q_1q_3 \\ -2q_1q_2 + 2q_4q_3 & q_4^2 - q_3^2 + q_1^2 - q_2^2 & 0 \\ -q_2^2 + q_1^2 - q_4^2 + q_3^2 & 2q_1q_2 + 2q_4q_3 & 0 \\ -2q_3q_2 - 2q_1q_4 & -2q_4q_2 + 2q_1q_3 & 0 \end{bmatrix} \begin{matrix} \left.\begin{matrix}1\\2\\3\end{matrix}\right\}\omega_x \\ \left.\begin{matrix}4\\5\\6\end{matrix}\right\}\omega_y \\ \left.\begin{matrix}7\\8\\9\end{matrix}\right\}\omega_z \end{matrix} \tag{89}$$

We only need to show that there always exist a selection of three rows which results in a full rank submatrix. We can start with rows $\{1, 5, 9\}$:

$$\texttt{det}\left(\mathfrak{Z}\{1,5,9\}\right) = \alpha\, q_1 q_2 q_3 q_4 \left(q_2^2 + q_1^2 + q_3^2 + q_4^2\right) = \alpha\, q_1 q_2 q_3 q_4 \tag{90}$$

This determinant is zero only when at least one component of $\bar{q}$ becomes zero. Thus we only need to investigate the cases where one or more components of $\bar{q}$ are zero. If only one component is zero, we can use the rows $\{1, 2, 4\}$, whose determinant is always nonzero if only one component of $\bar{q}$ is zero:

$$\mathtt{det}\left(\mathfrak{Z}\{1,2,4\}\right) = \alpha\ (q_1 q_2 - q_3 q_4)\,(-q_4 q_2 + q_1 q_3)\,\left(q_2^2 + q_1^2 + q_3^2 + q_4^2\right) = \alpha\ (q_1 q_2 - q_3 q_4)\,(-q_4 q_2 + q_1 q_3) \quad (91)$$

If exactly two components of $\bar{q}$ are zero, we can use the following selections of rows depending on which components are zero:

Zero Components	Selection	Determinant $\mathtt{det}\left(\mathfrak{Z}\{.\}\right)$
q_1 , q_2	{1,3,6}	$\alpha\,(q_2^2 + q_1^2 - q_4^2 - q_3^2)(q_1 q_2 + q_3 q_4)$
q_1 , q_3	{1,2,4}	$\alpha\,(q_1 q_3 + q_4 q_2)(q_1 q_3 - q_4 q_2)$
q_1 , q_4	{2,3,5}	$\alpha\,(-q_2 q_3 + q_1 q_4)(q_4^2 - q_3^2 + q_1^2 - q_2^2)$
q_2 , q_3	{2,3,5}	$\alpha\,(-q_3 q_2 + q_1 q_4)(q_4^2 - q_2^2 + q_1^2 - q_3^2)$
q_2 , q_4	{1,2,4}	$\alpha\,(q_1 q_3 + q_4 q_2)(-q_4 q_2 + q_1 q_3)$
q_3 , q_4	{1,3,6}	$\alpha\,(q_2^2 + q_1^2 - q_3^2 - q_4^2)(q_1 q_2 + q_3 q_4)$

Finally for the case when three components of $\bar{q}$ are zero, we can use selection $\{2, 3, 6\}$:

$$\begin{aligned}\mathtt{det}\left(\mathfrak{Z}\{2,3,6\}\right) &= \alpha\ \left(q_2^2 + q_1^2 - q_3^2 - q_4^2\right)\left(q_4^2 - q_2^2 + q_1^2 - q_3^2\right)\left(q_2^2 + q_1^2 + q_3^2 + q_4^2\right) \\ &= \alpha\ \left(q_2^2 + q_1^2 - q_3^2 - q_4^2\right)\left(q_4^2 - q_2^2 + q_1^2 - q_3^2\right) \end{aligned} \quad (92)$$

Thus the proof is complete. The MATLAB® script that was used for this proof is provided in Appendix C. □

Appendices

A SCRIPT 1

```
% SCRIPT 1

clc
clear

syms q_1 q_2 q_3 q_4
q = [q_1 q_2 q_3 q_4].';

syms e_1 e_2 e_3
e = [e_1 e_2 e_3].';

A1 = -2*q(4)*skewsymm(e) +...
    2*(skewsymm(e)*skewsymm(q(1:3)) - 2*skewsymm(q(1:3))*skewsymm(e))

A2 = 2*skewsymm(q(1:3))*e

A = [A1 A2];

B(1:3,:) = subs(A,[e_2 e_3], [0 0]);
B(4:6,:) = subs(A,[e_1 e_3], [0 0]);
B(7:9,:) = subs(A,[e_1 e_2], [0 0]);
B(10,:) = [q_1 q_2 q_3 q_4];

% define the symbols to be substituted or use empty to display all
% determinants.
vector = [e_3,q_1,q_2]
%vector = [];

subs_vect = zeros(size(vector));

for i = 1:10
    for j = i+1:10
        for k = j+1:10
            for l = k+1:10

                d = factor(det(B([i j k l],:)))

                if ¬isempty(vector)
                    res = subs(d,vector,subs_vect)

                    if res ≠ 0
                        [i,j,k,l]
                        keyboard%('non zero');
                    end
                else
                    [i,j,k,l]
                end
            end
        end
    end
end
```

B SCRIPT 2

```
% SCRIPT2

clc
clear

syms q_1 q_2 q_3 q_4
q = [q_1 q_2 q_3 q_4].';

syms s_1 s_2 s_3 s_4
s = [s_1 s_2 s_3 s_4].';

J = -eye(4);
J(4,4) = 1;

A = Right(q)*J*Xi(s)

d(1,1) = -factor(det(A([1 2 3],:)));
d(2,1) = -factor(det(A([1 2 4],:)));
d(3,1) = -factor(det(A([1 3 4],:)));
d(4,1) = factor(det(A([2 3 4],:)));

d1 = d/(s.'*s)/(q.'*q);

D(:,1) = diff(d1,s_1);
D(:,2) = diff(d1,s_2);
D(:,3) = diff(d1,s_3);
D(:,4) = diff(d1,s_4);

factor(det(D))
```

C SCRIPT 3

```
% SCRIPT3

clc
clear

syms q_1 q_2 q_3 q_4
q = [q_1 q_2 q_3 q_4].';

P{1} = [ 0 0 0 1 ; 0 0 1 0 ; 0 -1 0 0 ; -1 0 0 0]

P{2} = [ 0 0 -1 0 ; 0 0 0 1 ; 1 0 0 0 ; 0 -1 0 0]

P{3} = [ 0 1 0 0 ; -1 0 0 0 ; 0 0 0 1 ; 0 0 -1 0]


for i=1:3

    s = P{i}*q

    A(3*(i-1)+1:3*(i-1)+3,:) = -q(4)*skewsymm(s(1:3)) + skewsymm(skewsymm(q(1:3))*s(1:3)) - ...
        2*skewsymm(q(1:3))*skewsymm(s(1:3))-s(4)*skewsymm(q(1:3));

end

% "vector" contains the elements that you want zero out. If vector is
% empty, all determinant for all combinations of rows will be displayed.
vector = [q_2 q_4 q_3]
% vector = [];

subs_vect = zeros(size(vector));

for i = 1:9
    for j = i+1:9
        for k = j+1:9

            d = factor(det(A([i j k],:)))

                if ¬isempty(vector)
                    res = subs(d,vector,subs_vect)

                    if res ≠ 0
                        [i,j,k]
                        display('non zero');
                        return
                        % keyboard
                    end
                else
                    [i,j,k]
                end
        end
    end
end
```

D SCRIPT 4

```
% SCRIPT 4

clear
clc

syms p_1 p_2 p_3
p = [p_1 p_2 p_3].';
syms q_1 q_2 q_3 q_4
q = [q_1 q_2 q_3 q_4].';
qb = q(1:3);
syms e_1 e_2 e_3

A1 = -2*q_4*skewsymm(p) +...
    2*(skewsymm(p)*skewsymm(q(1:3)) - 2*skewsymm(q(1:3))*skewsymm(p))
A2 = 2*skewsymm(q(1:3))*p
A = [A1 A2];

e = diag([e_1 e_2 e_3]);
E = [];

for i = 1:3

    Lf1h2 = -0.5*A*Xi(q)*e(:,i)

    dLf1h2(:,1) = diff(Lf1h2,q_1);
    dLf1h2(:,2) = diff(Lf1h2,q_2);
    dLf1h2(:,3) = diff(Lf1h2,q_3);
    dLf1h2(:,4) = diff(Lf1h2,q_4);

    dLf1h2(:,5) = diff(Lf1h2,p_1);
    dLf1h2(:,6) = diff(Lf1h2,p_2);
    dLf1h2(:,7) = diff(Lf1h2,p_3)

    E = [E ; dLf1h2 ];
end

dL0h3 = [q_1 q_2 q_3 q_4 0 0 0];


% define the symbols to be substituted by one (e_1, e_2, or e_3)
one_subs = [];
one_vectors = ones(size(one_subs));
if ¬isempty(one_subs)
    E0 = subs(E,one_subs,one_subs);
else
    E0 = E;
end


% define the symbols to be substituted or use empty to display all
% determinants.
vector = [p_1]
%vector = [];
subs_vect = zeros(size(vector));

if ¬isempty(vector)
    E0 = subs(E0,vector,subs_vect);
end



for i1 = 1:9
    i1
    for i2 = i1+1:9
```

```
        for i3 = i2+1:9
            for i4 = i3+1:9
                for i5 = i4+1:9
                    for i6 = i5+1:9

                        res = factor(det([E0([i1 i2 i3 i4 i5 i6],:) ; dL0h3]));

                        if ¬isempty(vector)
                            if res ≠ 0
                                display('non zero');
                                [i1 i2 i3 i4 i5 i6]
                                res
                                % raw determinant (w/o subs)
                                factor(det([E([i1 i2 i3 i4 i5 i6],:) ; dL0h3]))
                                display('---------------------');
                                %return
                                keyboard
                            end
                        else
                            [i1 i2 i3 i4 i5 i6]
                        end

                    end
                end
            end
        end
    end
end

return
```

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