
Indirect Kalman Filter for 3D Attitude Estimation

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1 Elements of Quaternion Algebra

1.1 Quaternion Definitions

The quaternion is generally defined as

$$\bar{q} = q_4 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \quad (1)$$

where \mathbf{i} , \mathbf{j} , and \mathbf{k} are hyperimaginary numbers satisfying

$$\begin{aligned} \mathbf{i}^2 = -1, \quad \mathbf{j}^2 = -1, \quad \mathbf{k}^2 = -1, \\ -\mathbf{ij} = \mathbf{ji} = \mathbf{k}, \quad -\mathbf{jk} = \mathbf{kj} = \mathbf{i}, \quad -\mathbf{ki} = \mathbf{ik} = \mathbf{j} \end{aligned} \quad (2)$$

Note that this does not correspond to the Hamilton notation. It rather is a convention resulting in multiplications of quaternions in “natural order” (see also section 1.4 and [5, p. 473]). This is in accordance with the JPL Proposed Standard Conventions [2].

The quantity q_4 is the real or scalar part of the quaternion, and $q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ is the imaginary or vector part. The quaternion can therefore also be written in a four-dimensional column matrix \bar{q} , given by

$$\bar{q} = \begin{bmatrix} \mathbf{q} \\ q_4 \end{bmatrix} = [q_1 \quad q_2 \quad q_3 \quad q_4]^T \quad (3)$$

If the quantities \mathbf{q} and q_4 fulfill

$$\mathbf{q} = \begin{bmatrix} k_x \sin(\theta/2) \\ k_y \sin(\theta/2) \\ k_z \sin(\theta/2) \end{bmatrix} = \hat{\mathbf{k}} \sin(\theta/2), \quad q_4 = \cos(\theta/2) \quad (4)$$

the elements q_1, \dots, q_4 are called “quaternion of rotation” or “Euler symmetric parameters” [5]. In this notation, $\hat{\mathbf{k}}$ describes the unit vector along the axis and θ the angle of rotation. The quaternion of rotation is a unit quaternion, satisfying

$$|\bar{q}| = \sqrt{\bar{q}^T \bar{q}} = \sqrt{|\mathbf{q}|^2 + q_4^2} = 1 \quad (5)$$

Henceforth, we will use the term “quaternion” to refer to a quaternion of rotation.

The quaternion \bar{q} and the quaternion $-\bar{q}$ describe a rotation to the same final coordinate system position, i. e. the angle-axis representation is not unique [5, p. 463]. The only difference is the direction of rotation to get to the target configuration, with the quaternion with positive scalar element q_4 describing the shortest rotation [2].

1.2 Quaternion Multiplication

The quaternion multiplication is defined as

$$\begin{aligned}
\bar{q} \otimes \bar{p} &= (q_4 + q_1 \mathbf{i} + q_2 \mathbf{j} + q_3 \mathbf{k}) (p_4 + p_1 \mathbf{i} + p_2 \mathbf{j} + p_3 \mathbf{k}) \\
&= q_4 p_4 - q_1 p_1 - q_2 p_2 - q_3 p_3 + (q_4 p_1 + q_1 p_4 - q_2 p_3 + q_3 p_2) \mathbf{i} \\
&\quad + (q_4 p_2 + q_2 p_4 - q_3 p_1 + q_1 p_3) \mathbf{j} + (q_4 p_3 + q_3 p_4 - q_1 p_2 + q_2 p_1) \mathbf{k} \\
&= \begin{bmatrix} q_4 p_1 + q_3 p_2 - q_2 p_3 + q_1 p_4 \\ -q_3 p_1 + q_4 p_2 + q_1 p_3 + q_2 p_4 \\ q_2 p_1 - q_1 p_2 + q_4 p_3 + q_3 p_4 \\ -q_1 p_1 - q_2 p_2 - q_3 p_3 + q_4 p_4 \end{bmatrix}
\end{aligned}$$

where we have used the relations defined in Eq. (2).

The quaternion multiplication can alternatively be written in matrix form. For this, we first introduce the matrix-notation for the cross-product using the skew-symmetric matrix operator $[\mathbf{q} \times]$, defined as

$$[\mathbf{q} \times] = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix} \quad (6)$$

The cross product can then be written as

$$\mathbf{q} \times \mathbf{p} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ q_1 & q_2 & q_3 \\ p_1 & p_2 & p_3 \end{vmatrix} = \begin{bmatrix} q_2 p_3 - q_3 p_2 \\ q_3 p_1 - q_1 p_3 \\ q_1 p_2 - q_2 p_1 \end{bmatrix} = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = [\mathbf{q} \times] \mathbf{p} \quad (7)$$

The quaternion multiplication can now be rewritten in matrix form as

$$\begin{aligned}
\bar{q} \otimes \bar{p} &= \mathcal{L}(\bar{q}) \bar{p} \\
\bar{q} \otimes \bar{p} &= \begin{bmatrix} q_4 & q_3 & -q_2 & q_1 \\ -q_3 & q_4 & q_1 & q_2 \\ q_2 & -q_1 & q_4 & q_3 \\ -q_1 & -q_2 & -q_3 & q_4 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} \\
\bar{q} \otimes \bar{p} &= \begin{bmatrix} q_4 \mathbf{I}_{3 \times 3} - [\mathbf{q} \times] & \mathbf{q} \\ -\mathbf{q}^T & q_4 \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ p_4 \end{bmatrix} \quad (8) \\
&= \begin{bmatrix} q_4 \mathbf{p} + p_4 \mathbf{q} - \mathbf{q} \times \mathbf{p} \\ q_4 p_4 - \mathbf{q}^T \mathbf{p} \end{bmatrix} \\
&= \begin{bmatrix} p_4 \mathbf{q} + \mathbf{p} q_4 + [\mathbf{p} \times] \mathbf{q} \\ p_4 q_4 - \mathbf{p}^T \mathbf{q} \end{bmatrix} \quad (9)
\end{aligned}$$

$$\begin{aligned}
\bar{q} \otimes \bar{p} &= \begin{bmatrix} p_4 \mathbf{I}_{3 \times 3} + [\mathbf{p} \times] & \mathbf{p} \\ -\mathbf{p}^T & p_4 \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ q_4 \end{bmatrix} \\
\bar{q} \otimes \bar{p} &= \begin{bmatrix} p_4 & -p_3 & p_2 & p_1 \\ p_3 & p_4 & -p_1 & p_2 \\ -p_2 & p_1 & p_4 & p_3 \\ -p_1 & -p_2 & -p_3 & p_4 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \quad (10) \\
\bar{q} \otimes \bar{p} &= \mathcal{R}(\bar{p}) \bar{q}
\end{aligned}$$

Properties of \mathcal{L} and \mathcal{R}

$$\mathcal{L}(\bar{q}^{-1}) = \mathcal{L}^T(\bar{q}) \quad (11)$$

$$\mathcal{R}(\bar{p}^{-1}) = \mathcal{R}^T(\bar{p}) \quad (12)$$

$$(13)$$

$$\mathcal{L} = [\Psi(\bar{q}) \quad \bar{q}] \quad (14)$$

$$\mathcal{R} = [\Xi(\bar{p}) \quad \bar{p}] \quad (15)$$

$$(16)$$

with matrices Ψ and Ξ defined as

$$\Psi = \begin{bmatrix} q_4 \mathbf{I}_{3 \times 3} - [\mathbf{q} \times] \\ -\mathbf{q}^T \end{bmatrix} \quad (17)$$

$$\Xi = \begin{bmatrix} p_4 \mathbf{I}_{3 \times 3} + [\mathbf{p} \times] \\ -\mathbf{p}^T \end{bmatrix} \quad (18)$$

If two rotations, \mathbf{C}_A and \mathbf{C}_B are related, such that $\mathbf{C}_A \mathbf{C}_X = \mathbf{C}_X \mathbf{C}_B$ (hand-eye calibration), then the related matrix $\mathcal{L}(\bar{q}_A) - \mathcal{R}(\bar{q}_B)$ is skew-symmetric, and of rank 2. In particular, $\bar{q}_{A_4} = \bar{q}_{B_4}$ and $\|\mathbf{q}_A\| = \|\mathbf{q}_B\|$.

Quaternions also have a neutral element with respect to multiplication, which is defined as

$$\bar{q}_0 = [0 \quad 0 \quad 0 \quad 1]^T \quad (19)$$

$$\bar{q} \otimes \bar{q}_0 = \bar{q}_0 \otimes \bar{q} = \bar{q} \quad (20)$$

The inverse rotation is described by the inverse or complex conjugate quaternion, denoted as

$$\bar{q}^{-1} = \begin{bmatrix} -\mathbf{q} \\ q_4 \end{bmatrix} = \begin{bmatrix} -\hat{\mathbf{k}} \sin(\theta/2) \\ \cos(\theta/2) \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{k}} \sin(-\theta/2) \\ \cos(-\theta/2) \end{bmatrix} \quad (21)$$

$$\bar{q} \otimes \bar{q}^{-1} = \bar{q}^{-1} \otimes \bar{q} = \bar{q}_0 \quad (22)$$

$$(\bar{q} \otimes \bar{p})^{-1} = \bar{p}^{-1} \otimes \bar{q}^{-1} \quad (23)$$

1.3 Useful Identities

1.3.1 Properties of the cross product skew-symmetric matrix $[\boldsymbol{\omega} \times]$

Anti-Commutativity

$$[\boldsymbol{\omega} \times] = -[\boldsymbol{\omega} \times]^T \quad (24)$$

$$[\mathbf{a} \times] \mathbf{b} = -[\mathbf{b} \times] \mathbf{a} \quad (25)$$

$$\Leftrightarrow \mathbf{a}^T [\mathbf{b} \times] = -\mathbf{b}^T [\mathbf{a} \times] \quad (26)$$

Distributivity over Addition

$$[\mathbf{a} \times] + [\mathbf{b} \times] = [\mathbf{a} + \mathbf{b} \times] \quad (27)$$

Scalar Multiplication

$$c \cdot [\boldsymbol{\omega} \times] = [c \boldsymbol{\omega} \times] \quad (28)$$

Cross Product of Parallel Vectors

$$\boldsymbol{\omega} \times (c \cdot \boldsymbol{\omega}) = c \cdot [\boldsymbol{\omega} \times] \boldsymbol{\omega} = -c \cdot (\boldsymbol{\omega}^T [\boldsymbol{\omega} \times])^T = \mathbf{0}_{3 \times 1} \quad (29)$$

Lagrange's Formula

$$[\mathbf{a} \times][\mathbf{b} \times] = \mathbf{b}\mathbf{a}^T - (\mathbf{a}^T\mathbf{b})\mathbf{I}_{3 \times 3} \quad (30)$$

$$\Leftrightarrow \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a}^T\mathbf{c}) - \mathbf{c}(\mathbf{a}^T\mathbf{b}) \quad (31)$$

$$[\mathbf{a} \times][\mathbf{b} \times] + \mathbf{a}\mathbf{b}^T = [\mathbf{b} \times][\mathbf{a} \times] + \mathbf{b}\mathbf{a}^T \quad (32)$$

$$[(\mathbf{a} \times \mathbf{b}) \times] = \mathbf{b}\mathbf{a}^T - \mathbf{a}\mathbf{b}^T (= (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}) \quad (33)$$

Jacobi Identity

$$[\mathbf{a} \times][\mathbf{b} \times]\mathbf{c} + [\mathbf{b} \times][\mathbf{c} \times]\mathbf{a} + [\mathbf{c} \times][\mathbf{a} \times]\mathbf{b} = 0 \quad (34)$$

Rotations

$$[\mathbf{C}\mathbf{a} \times] = \mathbf{C}[\mathbf{a} \times]\mathbf{C}^T \quad (35)$$

$$\mathbf{C}(\mathbf{a} \times \mathbf{b}) = (\mathbf{C}\mathbf{a}) \times (\mathbf{C}\mathbf{b}) \quad (36)$$

Cross product of vectors in quaternion notation If we define the quaternions

$$\bar{a} = \begin{bmatrix} \mathbf{a} \\ 0 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix}, \quad \bar{c} = \begin{bmatrix} \mathbf{c} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{a} \times \mathbf{b} \\ 0 \end{bmatrix} \quad (37)$$

we can show that

$$\bar{c} = \frac{1}{2} (\bar{b} \otimes \bar{a} + \bar{a} \otimes \bar{b}^{-1}) \quad (38)$$

$$= \frac{1}{2} ((\mathcal{L}(\bar{b}) + \mathcal{R}^T(b))\bar{a}) \quad (39)$$

$$= \frac{1}{2} \left(\begin{bmatrix} -[\mathbf{b} \times] - [\mathbf{b} \times] & \mathbf{b} - \mathbf{b} \\ -\mathbf{b}^T + \mathbf{b}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ 0 \end{bmatrix} \right) \quad (40)$$

Powers of $[\omega \times]$

$$[\omega \times]^2 = \omega\omega^T - |\omega|^2 \cdot \mathbf{I}_{3 \times 3} \quad (41)$$

$$\begin{aligned} [\omega \times]^3 &= (\omega\omega^T - |\omega|^2 \cdot \mathbf{I}_{3 \times 3}) [\omega \times] \\ &= \omega\omega^T [\omega \times] - |\omega|^2 \cdot [\omega \times] \\ &= \omega (-[\omega \times] \omega)^T - |\omega|^2 \cdot [\omega \times] \\ &= -\omega (\omega \times \omega)^T - |\omega|^2 \cdot [\omega \times] \\ &= -|\omega|^2 \cdot [\omega \times] \end{aligned} \quad (42)$$

$$\begin{aligned} [\omega \times]^4 &= [\omega \times]^3 \cdot [\omega \times] \\ &= -|\omega|^2 \cdot [\omega \times]^2 \end{aligned} \quad (43)$$

$$\begin{aligned} [\omega \times]^5 &= [\omega \times]^3 \cdot [\omega \times]^2 \\ &= -|\omega|^2 \cdot [\omega \times] (\omega\omega^T - |\omega|^2 \cdot \mathbf{I}_{3 \times 3}) \\ &= +|\omega|^4 \cdot [\omega \times] \end{aligned} \quad (44)$$

$$\begin{aligned} [\omega \times]^6 &= [\omega \times]^5 \cdot [\omega \times] \\ &= +|\omega|^4 \cdot [\omega \times]^2 \end{aligned} \quad (45)$$

$$\begin{aligned} [\omega \times]^7 &= [\omega \times]^5 \cdot [\omega \times]^2 \\ &= -|\omega|^6 \cdot [\omega \times] \end{aligned} \quad (46)$$

and so on.

1.3.2 Properties of the matrix Ω

The matrix Ω appears in the product of a vector and a quaternion, and is used for example in the quaternion derivative. It has the following properties:

$$\Omega(\boldsymbol{\omega}) = \begin{bmatrix} 0 & \omega_z & -\omega_y & \omega_x \\ -\omega_z & 0 & \omega_x & \omega_y \\ \omega_y & -\omega_x & 0 & \omega_z \\ -\omega_x & -\omega_y & -\omega_z & 0 \end{bmatrix} \quad (47)$$

$$= \begin{bmatrix} -[\boldsymbol{\omega} \times] & \boldsymbol{\omega} \\ -\boldsymbol{\omega}^T & 0 \end{bmatrix} \quad (48)$$

$$\begin{aligned} \Omega(\boldsymbol{\omega})^2 &= \begin{bmatrix} [\boldsymbol{\omega} \times]^2 - \boldsymbol{\omega}\boldsymbol{\omega}^T & -[\boldsymbol{\omega} \times]\boldsymbol{\omega} \\ \boldsymbol{\omega}^T[\boldsymbol{\omega} \times] & -\boldsymbol{\omega}^T\boldsymbol{\omega} \end{bmatrix} \\ &= \begin{bmatrix} -|\boldsymbol{\omega}|^2 \cdot \mathbf{I}_{3 \times 3} & \mathbf{0}_{3 \times 1} \\ \mathbf{0}_{1 \times 3} & -|\boldsymbol{\omega}|^2 \end{bmatrix} \\ &= -|\boldsymbol{\omega}|^2 \cdot \mathbf{I}_{4 \times 4} \end{aligned} \quad (49)$$

$$\Omega(\boldsymbol{\omega})^3 = -|\boldsymbol{\omega}|^2 \cdot \Omega(\boldsymbol{\omega}) \quad (50)$$

$$\Omega(\boldsymbol{\omega})^4 = |\boldsymbol{\omega}|^4 \cdot \mathbf{I}_{4 \times 4} \quad (51)$$

$$\Omega(\boldsymbol{\omega})^5 = |\boldsymbol{\omega}|^4 \cdot \Omega(\boldsymbol{\omega}) \quad (52)$$

$$\Omega(\boldsymbol{\omega})^6 = -|\boldsymbol{\omega}|^6 \cdot \mathbf{I}_{4 \times 4} \quad (53)$$

and so on.

1.3.3 Properties of the matrix Ξ

The matrix $\Xi(\bar{q})$ appears in the multiplication of a vector with a quaternion. The relationship between $\Xi(\bar{q})$ and $\Omega(\mathbf{a})$ is equivalent to that between the multiplication matrices $\mathcal{L}(\bar{q})$ and $\mathcal{R}(\bar{p})$ (cf. section 1.2). It is defined as

$$\Xi(\bar{q}) = \begin{bmatrix} q_4 & -q_3 & q_2 \\ q_3 & q_4 & -q_1 \\ -q_2 & q_1 & q_4 \\ -q_1 & -q_2 & -q_3 \end{bmatrix} = \begin{bmatrix} q_4 \cdot \mathbf{I}_{3 \times 3} + [\mathbf{q} \times] \\ -\mathbf{q}^T \end{bmatrix}, \quad \Xi^T(\bar{q}) = \begin{bmatrix} q_4 & q_3 & -q_2 & -q_1 \\ -q_3 & q_4 & q_1 & -q_2 \\ q_2 & -q_1 & q_4 & -q_3 \end{bmatrix} \quad (54)$$

and it can be shown that

$$\Xi^T(\bar{q})\Xi(\bar{q}) = \mathbf{I}_{3 \times 3} \quad (55)$$

$$\Xi(\bar{q})\Xi^T(\bar{q}) = \mathbf{I}_{4 \times 4} - \bar{q}\bar{q}^T \quad (56)$$

$$\Xi^T(\bar{q})\bar{q} = \mathbf{0}_{3 \times 1} \quad (57)$$

The relationship between Ξ and Ω is given by [3, eq. (60)]

$$\Omega(\mathbf{a})\bar{q} = \Xi(\bar{q})\mathbf{a} \quad (58)$$

1.4 Relationship between Quaternion and Rotational Matrix

Given a vector \mathbf{p} we define the corresponding quaternion as

$$\bar{p} = \begin{bmatrix} \mathbf{p} \\ 0 \end{bmatrix} \quad (59)$$

We will be using the following two relations between vectors expressed in different coordinate frames

$${}^L\mathbf{p} = {}^L_G\mathbf{C}(\bar{q}) {}^G\mathbf{p} \quad (60)$$

where $\bar{q} = {}^L_G\bar{q}$ and ${}^L_G\mathbf{C}(\bar{q})$ is the (3×3) rotational matrix that expresses the (global) frame $\{G\}$ with respect to the (local) frame $\{L\}$.

A vector can also be transformed from one coordinate frame to another by pre- and postmultiplying its quaternion by the rotation quaternion and its inverse, respectively.

$$\begin{aligned} {}^L\bar{p} &= {}^L_G\bar{q} \otimes {}^G\bar{p} \otimes {}^L_G\bar{q}^{-1} \quad (61) \\ &= \begin{bmatrix} q_4\mathbf{I}_{3 \times 3} - [\mathbf{q} \times] & \mathbf{q} \\ -\mathbf{q}^T & q_4 \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ 0 \end{bmatrix} \otimes {}^L_G\bar{q}^{-1} \\ &= \begin{bmatrix} q_4\mathbf{p} - \mathbf{q} \times \mathbf{p} \\ -\mathbf{q}^T\mathbf{p} \end{bmatrix} \otimes \begin{bmatrix} -\mathbf{q} \\ q_4 \end{bmatrix} \\ &= \begin{bmatrix} q_4^2\mathbf{p} - q_4\mathbf{q} \times \mathbf{p} + \mathbf{q}\mathbf{q}^T\mathbf{p} + q_4\mathbf{p} \times \mathbf{q} - (\mathbf{q} \times \mathbf{p}) \times \mathbf{q} \\ +q_4\mathbf{q}^T\mathbf{p} - q_4\mathbf{q}^T\mathbf{p} - \mathbf{q}^T(\mathbf{q} \times \mathbf{p}) \end{bmatrix} \\ &= \begin{bmatrix} q_4^2\mathbf{p} - 2q_4\mathbf{q} \times \mathbf{p} + \mathbf{q}\mathbf{q}^T\mathbf{p} - ((1 - q_4^2)\mathbf{I}_{3 \times 3} - \mathbf{q}\mathbf{q}^T)\mathbf{p} \\ -\mathbf{q}^T[\mathbf{q} \times]\mathbf{p} \end{bmatrix} \\ &= \begin{bmatrix} (2q_4^2 - 1)\mathbf{I}_{3 \times 3} - 2q_4[\mathbf{q} \times] + 2\mathbf{q}\mathbf{q}^T \\ 0 \end{bmatrix} \begin{bmatrix} {}^G\mathbf{p} \\ 0 \end{bmatrix} \end{aligned}$$

This gives us the relationship between a quaternion and its corresponding rotational matrix.

$${}^L_G\mathbf{C}(\bar{q}) = (2q_4^2 - 1)\mathbf{I}_{3 \times 3} - 2q_4[\mathbf{q} \times] + 2\mathbf{q}\mathbf{q}^T \quad (62)$$

which can also be written as

$${}^L_G\mathbf{C}(\bar{q}) = \Xi^T(\bar{q})\Psi(\bar{q}) \quad (63)$$

using the definitions of Ξ and Ψ from Section 1.2.

A similar form can be derived for the triple product of quaternions, although without an obvious physical interpretation.

$$\bar{q} \otimes \bar{p} \otimes \bar{q}^{-1} \quad (64)$$

$$= \mathcal{L}(\bar{q})\mathcal{R}^T(\bar{q})\bar{p} \quad (65)$$

$$= \begin{bmatrix} \mathbf{C}(\bar{q}) & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ p_4 \end{bmatrix} \quad (66)$$

$$= \begin{bmatrix} \mathbf{C}(\bar{q})\mathbf{p} \\ p_4 \end{bmatrix} \quad (67)$$

In case of only a very small rotation $\delta\bar{q}$, we can use the small angle approximation to simplify the above expression. We can write the quaternion describing a small rotation as

$$\delta\bar{q} = \begin{bmatrix} \delta\mathbf{q} \\ \delta q_4 \end{bmatrix} \quad (68)$$

$$= \begin{bmatrix} \hat{\mathbf{k}} \sin(\delta\theta/2) \\ \cos(\delta\theta/2) \end{bmatrix} \quad (69)$$

$$\approx \begin{bmatrix} \frac{1}{2}\delta\boldsymbol{\theta} \\ 1 \end{bmatrix} \quad (70)$$

leading to the following expression for the corresponding rotational matrix

$${}^L_G\mathbf{C}(\delta\bar{q}) \approx \mathbf{I}_{3 \times 3} - [\delta\boldsymbol{\theta} \times] \quad (71)$$

The result from eq. (62) could have also been found by rewriting Euler's formula which relates the rotational matrix and the angle-axis representation [2]

$${}^L_G \mathbf{C} = \cos(\theta) \cdot \mathbf{I}_{3 \times 3} - \sin(\theta) [\hat{\mathbf{k}} \times] + (1 - \cos(\theta)) \hat{\mathbf{k}} \hat{\mathbf{k}}^T \quad (72)$$

$$= (2 \cos^2(\theta/2) - 1) \cdot \mathbf{I}_{3 \times 3} - 2 \cos(\theta/2) \sin(\theta/2) [\hat{\mathbf{k}} \times] + 2 \sin^2(\theta/2) \hat{\mathbf{k}} \hat{\mathbf{k}}^T \quad (73)$$

Substituting the appropriate quaternion components according to eq. (4) readily yields equation (62). The latter can also be expressed in terms of quaternion components as

$${}^L_G \mathbf{C}(\bar{q}) = \begin{bmatrix} q_1^2 - q_2^2 - q_3^2 + q_4^2 & 2(q_1 q_2 + q_3 q_4) & 2(q_1 q_3 - q_2 q_4) \\ 2(q_1 q_2 - q_3 q_4) & -q_1^2 + q_2^2 - q_3^2 + q_4^2 & 2(q_2 q_3 + q_1 q_4) \\ 2(q_1 q_3 + q_2 q_4) & 2(q_2 q_3 - q_1 q_4) & -q_1^2 - q_2^2 + q_3^2 + q_4^2 \end{bmatrix} \quad (74)$$

$$= \begin{bmatrix} 2q_1^2 + 2q_4^2 - 1 & 2(q_1 q_2 + q_3 q_4) & 2(q_1 q_3 - q_2 q_4) \\ 2(q_1 q_2 - q_3 q_4) & 2q_2^2 + 2q_4^2 - 1 & 2(q_2 q_3 + q_1 q_4) \\ 2(q_1 q_3 + q_2 q_4) & 2(q_2 q_3 - q_1 q_4) & 2q_3^2 + 2q_4^2 - 1 \end{bmatrix} \quad (75)$$

$$= \begin{bmatrix} 1 - 2q_2^2 - 2q_3^2 & 2(q_1 q_2 + q_3 q_4) & 2(q_1 q_3 - q_2 q_4) \\ 2(q_1 q_2 - q_3 q_4) & 1 - 2q_1^2 - 2q_3^2 & 2(q_2 q_3 + q_1 q_4) \\ 2(q_1 q_3 + q_2 q_4) & 2(q_2 q_3 - q_1 q_4) & 1 - 2q_1^2 - 2q_2^2 \end{bmatrix} \quad (76)$$

Another alternative form of eq. (62) arises after exploiting the relationship between $\mathbf{q} \mathbf{q}^T$ and $[\mathbf{q} \times]^2$ (cf. eq. (41))

$${}^L_G \mathbf{C}(\bar{q}) = \mathbf{I}_{3 \times 3} - 2q_4 [\mathbf{q} \times] + 2[\mathbf{q} \times]^2 \quad (77)$$

Note that, due to the convention chosen for the quaternion multiplication (cf. eq. (2)), the product of two rotational matrices will correspond to the product of two quaternions in the same order [2, 5]. Thus,

$${}^{L_1}_{L_2} \mathbf{C}({}^{L_1}_{L_2} \bar{q}) \cdot {}^{L_2}_G \mathbf{C}({}^{L_2}_G \bar{q}) = {}^{L_1}_G \mathbf{C} \left({}^{L_1}_{L_2} \bar{q} \otimes {}^{L_2}_G \bar{q} \right) \quad (78)$$

Finally, we can interpret ${}^{L_1}_{L_2} \bar{q} = \begin{bmatrix} \hat{\mathbf{k}} \sin(\theta/2) \\ \cos(\theta/2) \end{bmatrix}$ as the quaternion describing the rotation of coordinate frame $\{L_2\}$ to coordinate frame $\{L_1\}$, with the axis of rotation $\hat{\mathbf{k}}$ expressed in $\{L_1\}$ (cf. figure 1). This can also be expressed as the matrix exponential

$${}^L_G \mathbf{C}(\bar{q}) = \exp \left(-[\hat{\mathbf{k}} \times] \theta \right) \quad (79)$$

1.5 Quaternion Time Derivative

When the local coordinate frame $\{L\}$ is moving with respect to the global reference frame $\{G\}$, we can compute the rate of change or the derivative of the corresponding quaternion describing their relationship. We do that by computing the limit of the difference quotient

$${}^L_G \dot{\bar{q}}(t) = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left({}^{L(t+\Delta t)}_G \bar{q} - {}^{L(t)}_G \bar{q} \right) \quad (80)$$

The quaternion ${}^{L(t+\Delta t)}_G \bar{q}$ can be expressed as the product of two quaternions

$${}^L_G \bar{q} = {}^{L(t+\Delta t)}_{L(t)} \bar{q} \otimes {}^{L(t)}_G \bar{q} \quad (81)$$

where

$${}^{L(t+\Delta t)}_{L(t)} \bar{q} = \begin{bmatrix} \hat{\mathbf{k}} \sin(\theta/2) \\ \cos(\theta/2) \end{bmatrix} \quad (82)$$

Note, that the quaternion ${}^{L(t+\Delta t)}_{L(t)} \bar{q}$ describes the rotation of reference frame $\{L(t)\}$ to reference frame $\{L(t+\Delta t)\}$ in terms of the rotation angle θ and the axis of rotation $\hat{\mathbf{k}}$ (the latter being expressed in $\{L(t+\Delta t)\}$).

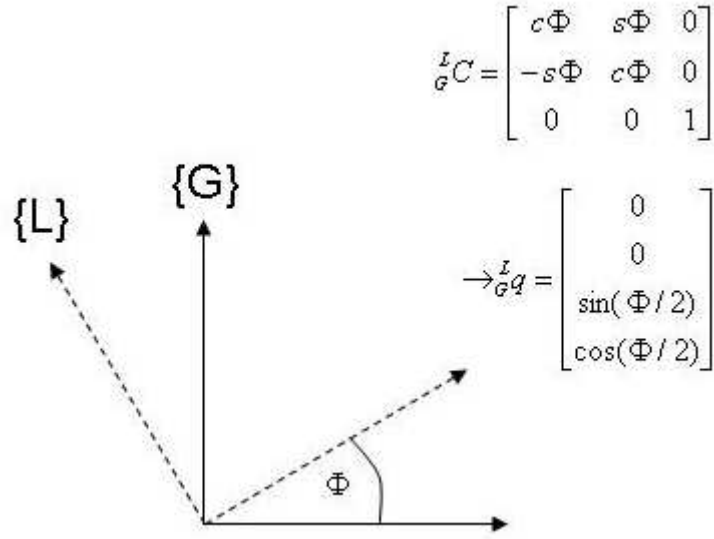


Figure 1: Relationship between global and local frame

In the limit, as $\Delta t \rightarrow 0$, the angle of rotation will become very small, so that we can approximate the sin and cos-functions by their first-order Taylor expansion

$$\begin{matrix} L(t+\Delta t) \\ L(t) \end{matrix} \bar{q} = \begin{bmatrix} \hat{\mathbf{k}} \sin(\theta/2) \\ \cos(\theta/2) \end{bmatrix} \approx \begin{bmatrix} \hat{\mathbf{k}} \cdot \theta/2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \cdot \delta\boldsymbol{\theta} \\ 1 \end{bmatrix} \quad (83)$$

The vector $\delta\boldsymbol{\theta}$ has the direction of the axis of rotation and the magnitude of the angle of the rotation. Dividing this vector by Δt will yield, in the limit, the rotational velocity

$$\boldsymbol{\omega} = \lim_{\Delta t \rightarrow 0} \frac{\delta\boldsymbol{\theta}}{\Delta t} \quad (84)$$

We are now ready to derive the quaternion derivative as

$$\begin{aligned} \begin{matrix} L(t) \\ G \end{matrix} \dot{\bar{q}}(t) &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left(\begin{matrix} L(t+\Delta t) \\ G \end{matrix} \bar{q} - \begin{matrix} L(t) \\ G \end{matrix} \bar{q} \right) \\ &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left(\begin{matrix} L(t+\Delta t) \\ L(t) \end{matrix} \bar{q} \otimes \begin{matrix} L(t) \\ G \end{matrix} \bar{q} - \bar{q}_0 \otimes \begin{matrix} L(t) \\ G \end{matrix} \bar{q} \right) \\ &\approx \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left(\begin{bmatrix} \frac{1}{2} \cdot \delta\boldsymbol{\theta} \\ 1 \end{bmatrix} - \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} \right) \otimes \begin{matrix} L(t) \\ G \end{matrix} \bar{q} \\ &= \frac{1}{2} \begin{bmatrix} \boldsymbol{\omega} \\ 0 \end{bmatrix} \otimes \begin{matrix} L(t) \\ G \end{matrix} \bar{q} \quad (85) \end{aligned}$$

$$= \frac{1}{2} \begin{bmatrix} -[\boldsymbol{\omega} \times] & \boldsymbol{\omega} \\ -\boldsymbol{\omega}^T & 0 \end{bmatrix} \begin{matrix} L(t) \\ G \end{matrix} \bar{q}$$

$$= \frac{1}{2} \boldsymbol{\Omega}(\boldsymbol{\omega}) \begin{matrix} L(t) \\ G \end{matrix} \bar{q} \quad (86)$$

$$= \frac{1}{2} \Xi \left(\begin{matrix} L(t) \\ G \end{matrix} \bar{q} \right) \boldsymbol{\omega} \quad (87)$$

with

$$\boldsymbol{\Omega}(\boldsymbol{\omega}) = \begin{bmatrix} 0 & \omega_z & -\omega_y & \omega_x \\ -\omega_z & 0 & \omega_x & \omega_y \\ \omega_y & -\omega_x & 0 & \omega_z \\ -\omega_x & -\omega_y & -\omega_z & 0 \end{bmatrix} \quad (88)$$

and (cf. sections 1.3.2 and 1.3.3)

$$\Xi_G^{(L(t)\bar{q})} = \begin{bmatrix} q_4 & -q_3 & q_2 \\ q_3 & q_4 & -q_1 \\ -q_2 & q_1 & q_4 \\ -q_1 & -q_2 & -q_3 \end{bmatrix} \quad (89)$$

Note that $\omega = {}^{L(t)}\omega$, i. e. the turn rate is expressed in the local, not the inertial coordinate frame [5, p. 482].

1.6 Quaternion Integration

Integrating a quaternion is equivalent to solving the first order differential equation (cf. eq. (86))

$${}^L_G \dot{\bar{q}}(t) = \frac{1}{2} \Omega(\omega) {}^L_G \bar{q}(t) \quad (90)$$

where we have dropped the time index from the leading superscript and instead denoted the time variability of the quaternion by writing $\bar{q} = \bar{q}(t)$ for greater notational clarity. It should be clear that the leading superscript L refers to the local frame $\{L\}$ at time instant t .

The solution to this differential equation has the general form [4, p. 40]

$${}^L_G \bar{q}(t) = \Theta(t, t_k) {}^L_G \bar{q}(t_k) \quad (91)$$

Differentiating and reordering the terms yields the governing equation for $\Theta(t, t_k)$ as

$$\dot{\Theta}(t, t_k) = {}^L_G \dot{\bar{q}}(t) {}^L_G \bar{q}^{-1}(t_k) \quad (92)$$

$$= \frac{1}{2} \Omega(\omega(t)) {}^L_G \bar{q}(t) {}^L_G \bar{q}^{-1}(t_k) \quad (93)$$

$$= \frac{1}{2} \Omega(\omega(t)) \Theta(t, t_k) \quad (94)$$

with the initial condition

$$\Theta(t_k, t_k) = \mathbf{I}_{4 \times 4} \quad (95)$$

Under certain assumptions we can obtain closed form solutions for this equation. The simplest assumption is that ω is constant over the integration period $\Delta t = t_{k+1} - t_k$, thus making the differential equation linear time invariant. This assumption leads to the zeroth order quaternion integrator. The next more accurate approximation is to assume a linear evolution of ω during Δt . We will term the resulting formula the first order quaternion integrator.

1.6.1 Zeroth Order Quaternion Integrator

If $\omega(t) = \omega$ is constant over the integration period $\Delta t = t_{k+1} - t_k$, the matrix Ω does not depend on time, and $\Theta(t_{k+1}, t_k)$ can be expressed as [4, eq. (2-58a)]

$$\Theta(t_{k+1}, t_k) = \Theta(\Delta t) = \exp\left(\frac{1}{2} \Omega(\omega) \Delta t\right) \quad (96)$$

We can rewrite this matrix exponential using its Taylor series expansion

$$\Theta(\Delta t) = \mathbf{I}_{4 \times 4} + \frac{1}{2} \Omega(\omega) \Delta t + \frac{1}{2!} \left(\frac{1}{2} \Omega(\omega) \Delta t\right)^2 + \frac{1}{3!} \left(\frac{1}{2} \Omega(\omega) \Delta t\right)^3 + \dots \quad (97)$$

Using the properties of the matrix Ω described in section 1.3.2, this transforms into

$$\begin{aligned} \Theta(\Delta t) &= \mathbf{I}_{4 \times 4} + \frac{1}{2} \Delta t \Omega(\omega) - \frac{1}{2!} \left(\frac{1}{2} \Delta t\right)^2 |\omega|^2 \cdot \mathbf{I}_{4 \times 4} \\ &\quad - \frac{1}{3!} \left(\frac{1}{2} \Delta t\right)^3 |\omega|^2 \cdot \Omega(\omega) + \frac{1}{4!} \left(\frac{1}{2} \Delta t\right)^4 |\omega|^4 \cdot \mathbf{I}_{4 \times 4} \\ &\quad + \frac{1}{5!} \left(\frac{1}{2} \Delta t\right)^5 |\omega|^4 \cdot \Omega(\omega) - \frac{1}{6!} \left(\frac{1}{2} \Delta t\right)^6 |\omega|^6 \cdot \mathbf{I}_{4 \times 4} - \dots \end{aligned} \quad (98)$$

Reordering and expanding with $|\omega|$ yields

$$\begin{aligned}\Theta(\Delta t) &= \left(1 - \frac{1}{2!} \left(\frac{1}{2}\Delta t\right)^2 |\omega|^2 + \frac{1}{4!} \left(\frac{1}{2}\Delta t\right)^4 |\omega|^4 - \dots\right) \mathbf{I}_{4 \times 4} \\ &+ \frac{1}{|\omega|} \left(\frac{1}{2}|\omega|\Delta t - \frac{1}{3!} \left(\frac{1}{2}|\omega|\Delta t\right)^3 + \frac{1}{5!} \left(\frac{1}{2}|\omega|\Delta t\right)^5 - \dots\right) \Omega(\omega)\end{aligned}\quad (99)$$

After close inspection, we recognize the Taylor series expansions of the sin and cos functions

$$\Theta(\Delta t) = \cos\left(\frac{|\omega|}{2}\Delta t\right) \cdot \mathbf{I}_{4 \times 4} + \frac{1}{|\omega|} \sin\left(\frac{|\omega|}{2}\Delta t\right) \cdot \Omega(\omega)\quad (100)$$

We can finally write the zero-th order quaternion integrator as

$$\begin{aligned}{}^L_G\bar{q}(t_{k+1}) &= \Theta(t_{k+1}, t_k) {}^L_G\bar{q}(t_k) \\ &= \left(\cos\left(\frac{|\omega|}{2}\Delta t\right) \cdot \mathbf{I}_{4 \times 4} + \frac{1}{|\omega|} \sin\left(\frac{|\omega|}{2}\Delta t\right) \cdot \Omega(\omega)\right) {}^L_G\bar{q}(t_k)\end{aligned}\quad (101)$$

A close look reveals that $\Theta(t_{k+1}, t_k)$ is nothing but the multiplication matrix associated with a specific quaternion, so that we can rewrite the zero-th order quaternion integration as a quaternion product

$${}^L_G\bar{q}(t_{k+1}) = \begin{bmatrix} \frac{\omega}{|\omega|} \cdot \sin\left(\frac{|\omega|}{2}\Delta t\right) \\ \cos\left(\frac{|\omega|}{2}\Delta t\right) \end{bmatrix} \otimes {}^L_G\bar{q}(t_k)\quad (102)$$

This quaternion product corresponds to rotating the original frame around the rotation axis defined by ω through the angle $|\omega|\Delta t$, which corresponds exactly to the assumption of a constant ω .

The above expression will cause numerical instability for very small ω , due to $|\omega|$ appearing in the denominator. We will therefore compute the limit of the above equation as $|\omega|$ goes towards zero, making multiple use of L'Hôpital's rule.

$$\begin{aligned}\lim_{|\omega| \rightarrow 0} \Theta(\Delta t) &= \lim_{|\omega| \rightarrow 0} \left(\cos\left(\frac{|\omega|}{2}\Delta t\right) \mathbf{I}_{4 \times 4} + \frac{1}{|\omega|} \sin\left(\frac{|\omega|}{2}\Delta t\right) \Omega(\omega)\right) \\ &= \mathbf{I}_{4 \times 4} + \lim_{|\omega| \rightarrow 0} \left(\frac{1}{|\omega|} \sin\left(\frac{|\omega|}{2}\Delta t\right) \Omega(\omega)\right) \\ &= \mathbf{I}_{4 \times 4} + \frac{\Delta t}{2} \Omega(\omega)\end{aligned}\quad (103)$$

1.6.2 First Order Quaternion Integrator

The first order quaternion integrator makes the assumption of a linear evolution of ω during the integration interval Δt . In this case, we have to modify the matrix $\Theta(t_{k+1}, t_k)$ from eq. (96). For that purpose, we introduce the average turn rate $\bar{\omega}$, defined as

$$\bar{\omega} = \frac{\omega(t_{k+1}) + \omega(t_k)}{2}\quad (104)$$

We can also define the derivative of the turn rate $\dot{\omega}$ and the associated matrix $\Omega(\dot{\omega})$, which, in the linear case, is constant

$$\Omega(\dot{\omega}) = \Omega\left(\frac{\omega(t_{k+1}) - \omega(t_k)}{\Delta t}\right)\quad (105)$$

Note that higher order derivatives of $\Omega(\omega)$ are zero.

Following Wertz [8, p. 565], in order to compute the quaternion at time instant t_{k+1} , we write its Taylor series expansion around time instant t_k

$${}^L_G\bar{q}(t_{k+1}) = {}^L_G\bar{q}(t_k) + {}^L_G\dot{\bar{q}}(t_k)\Delta t + \frac{1}{2} {}^L_G\ddot{\bar{q}}(t_k)\Delta t^2 + \dots\quad (106)$$

Repeatedly applying the definition of the quaternion time derivative (eq. (86)) yields

$$\begin{aligned} {}^L_G\bar{q}(t_{k+1}) &= \left(\mathbf{I}_{4 \times 4} + \frac{1}{2} \boldsymbol{\Omega}(\boldsymbol{\omega}(t_k))\Delta t + \frac{1}{2!} \left(\frac{1}{2} \boldsymbol{\Omega}(\boldsymbol{\omega}(t_k))\Delta t \right)^2 + \frac{1}{3!} \left(\frac{1}{2} \boldsymbol{\Omega}(\boldsymbol{\omega}(t_k))\Delta t \right)^3 + \dots \right) {}^L_G\bar{q}(t_k) \\ &+ \frac{1}{4} \Delta t^2 \boldsymbol{\Omega}(\dot{\boldsymbol{\omega}}(t_k)) {}^L_G\bar{q}(t_k) + \left(\frac{1}{12} \boldsymbol{\Omega}(\dot{\boldsymbol{\omega}}(t_k))\boldsymbol{\Omega}(\boldsymbol{\omega}(t_k)) + \frac{1}{24} \boldsymbol{\Omega}(\boldsymbol{\omega}(t_k))\boldsymbol{\Omega}(\dot{\boldsymbol{\omega}}(t_k)) \right) \Delta t^3 {}^L_G\bar{q}(t_k) + \dots \end{aligned} \quad (107)$$

If we write the average $\boldsymbol{\Omega}(\bar{\boldsymbol{\omega}})$ as

$$\boldsymbol{\Omega}(\bar{\boldsymbol{\omega}}) = \frac{1}{\Delta t} \int_{t_k}^{t_{k+1}} \boldsymbol{\Omega}(\boldsymbol{\omega}(\tau)) d\tau = \boldsymbol{\Omega}(\boldsymbol{\omega}(t_k)) + \frac{1}{2} \boldsymbol{\Omega}(\dot{\boldsymbol{\omega}}(t_k))\Delta t \quad (108)$$

we can reorder the terms in eq. (107) to form

$$\begin{aligned} {}^L_G\bar{q}(t_{k+1}) &= \left(\mathbf{I}_{4 \times 4} + \frac{1}{2} \boldsymbol{\Omega}(\bar{\boldsymbol{\omega}})\Delta t + \frac{1}{2!} \left(\frac{1}{2} \boldsymbol{\Omega}(\bar{\boldsymbol{\omega}})\Delta t \right)^2 + \frac{1}{3!} \left(\frac{1}{2} \boldsymbol{\Omega}(\bar{\boldsymbol{\omega}})\Delta t \right)^3 + \dots \right) \\ &+ \frac{1}{48} \left(\boldsymbol{\Omega}(\dot{\boldsymbol{\omega}}(t_k))\boldsymbol{\Omega}(\boldsymbol{\omega}(t_k)) - \boldsymbol{\Omega}(\boldsymbol{\omega}(t_k))\boldsymbol{\Omega}(\dot{\boldsymbol{\omega}}(t_k)) \right) \Delta t^3 {}^L_G\bar{q}(t_k) \end{aligned} \quad (109)$$

Recognizing the first term as the Taylor series expansion of the matrix exponential, and after replacing $\boldsymbol{\Omega}(\dot{\boldsymbol{\omega}})$ with its definition (eq. (105)), we obtain the final formula

$${}^L_G\bar{q}(t_{k+1}) = \left(\exp \left(\frac{1}{2} \boldsymbol{\Omega}(\bar{\boldsymbol{\omega}})\Delta t \right) + \frac{1}{48} \left(\boldsymbol{\Omega}(\boldsymbol{\omega}(t_{k+1}))\boldsymbol{\Omega}(\boldsymbol{\omega}(t_k)) - \boldsymbol{\Omega}(\boldsymbol{\omega}(t_k))\boldsymbol{\Omega}(\boldsymbol{\omega}(t_{k+1})) \right) \Delta t^2 \right) {}^L_G\bar{q}(t_k) \quad (110)$$

2 Attitude Propagation

The Kalman Filter consists of two stages to determine an estimate of the current attitude. In the first stage, the propagation, the filter produces a prediction of the attitude based on the last estimate and some proprioceptive measurements. This estimate is then corrected in the update stage, where new absolute orientation measurements are taken into account.

One way to predict the system state is to feed the control commands given to the system into a system model and thus to predict the behavior of the system. An alternative way to estimate position and orientation is to use data from an inertial measurement unit (IMU) as dynamic model replacement. The IMU provides measurements of the translational accelerations and rotational velocities acting on the system. In this paper, we will pursue the latter approach.

Before discussing the state equation and the error propagation, we will describe the model for the gyros that will provide measurements of the rotational velocity.

2.1 Gyro Noise Model

As part of the IMU, a three-axis gyro provides measurements of the rotational velocity. Gyros are known to be subject to different error terms, such as a rate noise error and a bias. In accordance with the literature [3, 1], we use a simple model that relates the measured turn rate ω_m to the real angular velocity ω as

$$\omega_m = \omega + \mathbf{b} + \mathbf{n}_r \quad (111)$$

In this equation, \mathbf{b} denotes the gyro bias and \mathbf{n}_r the rate noise, assumed to be Gaussian white noise with characteristics

$$E[\mathbf{n}_r] = \mathbf{0}_{3 \times 1} \quad (112)$$

$$E[\mathbf{n}_r(t + \tau)\mathbf{n}_r^T(t)] = \mathbf{N}_r\delta(\tau) \quad (113)$$

The gyro bias is non-static and simulated as a random walk process

$$\dot{\mathbf{b}} = \mathbf{n}_w \quad (114)$$

with characteristics

$$E[\mathbf{n}_w] = \mathbf{0}_{3 \times 1} \quad (115)$$

$$E[\mathbf{n}_w(t + \tau)\mathbf{n}_w^T(t)] = \mathbf{N}_w\delta(\tau) \quad (116)$$

The bias is therefore a random quantity and needs to be estimated along with the quaternion.

For simplification we will assume that the noise is equal in all three spatial directions, i. e.

$$\mathbf{N}_r = \sigma_{r_c}^2 \cdot \mathbf{I}_{3 \times 3} \quad (117)$$

$$\mathbf{N}_w = \sigma_{w_c}^2 \cdot \mathbf{I}_{3 \times 3} \quad (118)$$

The subscript c indicates, that these are the noise covariances for the continuous-time system, distinguishing them from the noise covariances in the discretized system used later.

In order to determine the units of the covariances [1] we consider the scalar case for illustration purposes, which extend directly to the vector case. For compatibility, n_r has the same units as ω , therefore

$$\left[E[n_r(t + \tau)n_r(t)] \right] = [\sigma_{r_c}^2 \delta(\tau)] = \frac{\text{rad}^2}{\text{sec}^2} \quad (119)$$

But the unit of $\delta(\tau)$ is defined as [4, p. 221]

$$[\delta(\tau)] = \frac{1}{\text{sec}} \quad (120)$$

Therefore

$$[\sigma_{r_c}^2] = \frac{\text{rad}^2}{\text{sec}^2} \cdot \text{sec} \quad (121)$$

$$= \left(\frac{\text{rad}}{\text{sec}} \right)^2 \cdot \frac{1}{\frac{1}{\text{sec}}} \quad (122)$$

$$= \left(\frac{\text{rad}}{\text{sec}} \right)^2 \cdot \frac{1}{\text{Hz}} \quad (123)$$

and

$$[\sigma_{r_c}] = \left(\frac{\text{rad}}{\text{sec}} \right) \cdot \frac{1}{\sqrt{\text{Hz}}} = \frac{\text{rad}}{\sqrt{\text{sec}}} \quad (124)$$

Analogously, we obtain

$$[\sigma_{w_c}^2] = \left(\frac{\text{rad}}{\text{sec}} / \text{sec} \right)^2 \cdot \text{sec} \quad (125)$$

$$= \frac{\text{rad}^2}{\text{sec}^3} \quad (126)$$

$$= \left(\frac{\text{rad}}{\text{sec}} \right)^2 \cdot \text{Hz} \quad (127)$$

and

$$[\sigma_{w_c}] = \left(\frac{\text{rad}}{\text{sec}} \right) \cdot \sqrt{\text{Hz}} = \frac{\text{rad}}{\sqrt{\text{sec}^3}} \quad (128)$$

In the discrete case, \mathbf{n}_{r_d} and \mathbf{n}_{w_d} will have to have the same units as the turn rate (cf. also the state equation in section 2.2), namely $\frac{\text{rad}}{\text{sec}}$. In order to preserve equivalence of the noise strength, we have to incorporate the sampling frequency when discretizing, yielding

$$\sigma_{r_d} = \frac{\sigma_{r_c}}{\sqrt{\Delta t}} \quad (129)$$

$$\sigma_{w_d} = \sigma_{w_c} \cdot \sqrt{\Delta t} \quad (130)$$

where Δt is the inverse of the sampling frequency, or $f_{\text{sample}} = \frac{1}{\Delta t}$. The discrete variances will be used in the simulation of noisy real-world data.

Further explanation for conversion between continuous and discrete noise variance is provided by Simon [6, pp. 230-233]. To make the analogy to the derivation in the book, the bias driving noise, \mathbf{n}_w , corresponds to the process noise, whereas the rate noise, \mathbf{n}_r , corresponds to the measurement noise.

2.2 The State Equation

In direct consequence of the previous analysis, we define a seven-element state vector consisting of the quaternion and the gyro-bias

$$\mathbf{x}(t) = \begin{bmatrix} \bar{q}(t) \\ \mathbf{b}(t) \end{bmatrix} \quad (131)$$

Using the definition of the quaternion derivative (eq. (86)) and the error model (eqs. (111) and (114)), we find the following system of differential equations governing the state

$${}^L_G \dot{\bar{q}}(t) = \frac{1}{2} \boldsymbol{\Omega}(\boldsymbol{\omega}_m - \mathbf{b} - \mathbf{n}_r) {}^L_G \bar{q}(t) \quad (132)$$

$$\dot{\mathbf{b}} = \mathbf{n}_w \quad (133)$$

Taking the expectation of the above yields the prediction equations (cf. [3, p. 422]) for the state within the EKF-framework

$${}^L_G \dot{\hat{q}}(t) = \frac{1}{2} \boldsymbol{\Omega}(\hat{\boldsymbol{\omega}}) {}^L_G \hat{q}(t) \quad (134)$$

$$\dot{\hat{\mathbf{b}}} = \mathbf{0}_{3 \times 1} \quad (135)$$

with

$$\hat{\boldsymbol{\omega}} = \boldsymbol{\omega}_m - \hat{\mathbf{b}} \quad (136)$$

Since the bias is constant over the integration interval, we may integrate the quaternion using the zeroth order (cf. section 1.6.1) or first order (cf. section 1.6.2) integrator, using $\hat{\boldsymbol{\omega}}$ instead of $\boldsymbol{\omega}$.

2.3 Error and Covariance Representation

Usually, the error vector and its covariance is expressed in terms of the arithmetic difference between the state vector and its estimate. In the problem at hand, however, this representation is problematic, due to the presence of constraints in the system. The fact that the quaternion is enforced to be of unit length (cf. eq. (5)) makes the corresponding covariance matrix singular [3, p. 423], which is difficult to maintain numerically. For stability reasons, we will therefore use a different, six-dimensional representation of the error vector.

Instead of using the arithmetic difference between quaternion and quaternion estimate to define the error, we will employ the error quaternion $\delta\hat{q}$; a small rotation between the estimated and the true orientation of the local frame of reference. Instead of a difference, this error is defined as a multiplication.

$${}^L_G\bar{q} = {}^L_{\hat{L}}\delta\bar{q} \otimes {}^{\hat{L}}_{\hat{G}}\hat{q} \quad (137)$$

$${}^L_{\hat{L}}\delta\bar{q} = {}^L_G\bar{q} \otimes {}^{\hat{L}}_{\hat{G}}\hat{q}^{-1} \quad (138)$$

Since the rotation associated with the error quaternion $\delta\bar{q}$ can be assumed to be very small, we can employ the small angle approximation (as seen in section 1.4) and define the attitude error angle vector $\delta\theta$ as follows

$$\delta\bar{q} = \begin{bmatrix} \delta\mathbf{q} \\ \delta q_4 \end{bmatrix} \quad (139)$$

$$= \begin{bmatrix} \hat{\mathbf{k}} \sin(\delta\theta/2) \\ \cos(\delta\theta/2) \end{bmatrix} \quad (140)$$

$$\approx \begin{bmatrix} \frac{1}{2}\delta\theta \\ 1 \end{bmatrix} \quad (141)$$

This error angle vector $\delta\theta$ is of dimension 3×1 and will be used together with the bias error in the error state vector. The bias error is defined as

$$\Delta\mathbf{b} = \mathbf{b} - \hat{\mathbf{b}} \quad (142)$$

We can now define the error vector as

$$\tilde{\mathbf{x}} = \begin{bmatrix} \delta\theta \\ \Delta\mathbf{b} \end{bmatrix} \quad (143)$$

In the next section we will develop the continuous time first order state equation for the error vector.

2.4 Continuous Time Error State Equations

In order to derive the continuous time linear state equations for the error vector, we will start with the definition of the error quaternion (eq. (137))

$$\bar{q} = \delta\bar{q} \otimes \hat{q} \quad \left| \frac{d}{dt} \right. \quad (144)$$

$$\dot{\bar{q}} = \dot{\delta\bar{q}} \otimes \hat{q} + \delta\bar{q} \otimes \dot{\hat{q}} \quad (145)$$

Substituting the definitions for $\dot{\hat{q}}$ (eq. (85)) and $\dot{\hat{q}}$ (eq. (134)) leads to

$$\frac{1}{2} \begin{bmatrix} \boldsymbol{\omega} \\ 0 \end{bmatrix} \otimes \bar{q} = \dot{\delta\bar{q}} \otimes \hat{q} + \delta\bar{q} \otimes \left(\frac{1}{2} \begin{bmatrix} \hat{\boldsymbol{\omega}} \\ 0 \end{bmatrix} \otimes \hat{q} \right) \quad \left| - \frac{1}{2}(\delta\bar{q} \otimes \begin{bmatrix} \hat{\boldsymbol{\omega}} \\ 0 \end{bmatrix} \otimes \hat{q}) \right. \quad (146)$$

$$\dot{\delta\bar{q}} \otimes \hat{q} = \frac{1}{2} \left(\begin{bmatrix} \boldsymbol{\omega} \\ 0 \end{bmatrix} \otimes \bar{q} - \delta\bar{q} \otimes \begin{bmatrix} \hat{\boldsymbol{\omega}} \\ 0 \end{bmatrix} \otimes \hat{q} \right) \quad \left| \otimes \hat{q}^{-1}, \text{ eq. (138)} \right. \quad (147)$$

$$\dot{\delta\bar{q}} = \frac{1}{2} \left(\begin{bmatrix} \boldsymbol{\omega} \\ 0 \end{bmatrix} \otimes \delta\bar{q} - \delta\bar{q} \otimes \begin{bmatrix} \hat{\boldsymbol{\omega}} \\ 0 \end{bmatrix} \right) \quad (148)$$

Combining the gyro model for $\boldsymbol{\omega}$ (cf. eq. (111)) and the definition of $\hat{\boldsymbol{\omega}}$ (cf. eq. (136)) yields

$$\boldsymbol{\omega} = \hat{\boldsymbol{\omega}} - \Delta\mathbf{b} - \mathbf{n}_r \quad (149)$$

which, upon substitution into the above leads to

$$\delta \dot{\bar{q}} = \frac{1}{2} \left(\begin{bmatrix} \hat{\boldsymbol{\omega}} \\ 0 \end{bmatrix} \otimes \delta \bar{q} - \delta \bar{q} \otimes \begin{bmatrix} \hat{\boldsymbol{\omega}} \\ 0 \end{bmatrix} \right) - \frac{1}{2} \begin{bmatrix} \Delta \mathbf{b} + \mathbf{n}_r \\ 0 \end{bmatrix} \otimes \delta \bar{q} \quad (150)$$

$$= \frac{1}{2} \left(\begin{bmatrix} -[\hat{\boldsymbol{\omega}} \times] & \hat{\boldsymbol{\omega}} \\ -\hat{\boldsymbol{\omega}}^T & 0 \end{bmatrix} \cdot \delta \bar{q} - \begin{bmatrix} +[\hat{\boldsymbol{\omega}} \times] & \hat{\boldsymbol{\omega}} \\ -\hat{\boldsymbol{\omega}}^T & 0 \end{bmatrix} \cdot \delta \bar{q} \right) - \frac{1}{2} \begin{bmatrix} \Delta \mathbf{b} + \mathbf{n}_r \\ 0 \end{bmatrix} \otimes \delta \bar{q} \quad (151)$$

$$= \frac{1}{2} \begin{bmatrix} -2[\hat{\boldsymbol{\omega}} \times] & \mathbf{0}_{3 \times 1} \\ \mathbf{0}_{3 \times 1}^T & 0 \end{bmatrix} \cdot \delta \bar{q} - \frac{1}{2} \begin{bmatrix} \Delta \mathbf{b} + \mathbf{n}_r \\ 0 \end{bmatrix} \otimes \delta \bar{q} \quad (152)$$

$$= \frac{1}{2} \begin{bmatrix} -2[\hat{\boldsymbol{\omega}} \times] & \mathbf{0}_{3 \times 1} \\ \mathbf{0}_{3 \times 1}^T & 0 \end{bmatrix} \cdot \delta \bar{q} - \frac{1}{2} \begin{bmatrix} -[(\Delta \mathbf{b} + \mathbf{n}_r) \times] & (\Delta \mathbf{b} + \mathbf{n}_r) \\ -(\Delta \mathbf{b} + \mathbf{n}_r)^T & 0 \end{bmatrix} \cdot \begin{bmatrix} \delta \mathbf{q} \\ 1 \end{bmatrix} \quad (153)$$

$$= \frac{1}{2} \begin{bmatrix} -2[\hat{\boldsymbol{\omega}} \times] & \mathbf{0}_{3 \times 1} \\ \mathbf{0}_{3 \times 1}^T & 0 \end{bmatrix} \cdot \delta \bar{q} - \frac{1}{2} \begin{bmatrix} (\Delta \mathbf{b} + \mathbf{n}_r) \\ 0 \end{bmatrix} - O(|\Delta \mathbf{b}| |\delta \mathbf{q}|, |\mathbf{n}_r| |\delta \mathbf{q}|) \quad (154)$$

Neglecting the second order terms, we can write

$$\delta \dot{\bar{q}} = \begin{bmatrix} \delta \dot{\mathbf{q}} \\ \delta \dot{q}_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \delta \dot{\boldsymbol{\theta}} \\ \dot{1} \end{bmatrix} = \begin{bmatrix} -\hat{\boldsymbol{\omega}} \times \delta \mathbf{q} - \frac{1}{2} (\Delta \mathbf{b} + \mathbf{n}_r) \\ 0 \end{bmatrix} \quad (155)$$

or finally

$$\delta \dot{\boldsymbol{\theta}} = -\hat{\boldsymbol{\omega}} \times \delta \boldsymbol{\theta} - \Delta \mathbf{b} - \mathbf{n}_r \quad (156)$$

The governing equation for the bias error is easily computed from eqs. (133) and (135) as

$$\Delta \dot{\mathbf{b}} = \dot{\mathbf{b}} - \hat{\mathbf{b}} = \mathbf{n}_w \quad (157)$$

Combining these results, we may write the error state equations as

$$\begin{bmatrix} \delta \dot{\boldsymbol{\theta}} \\ \Delta \dot{\mathbf{b}} \end{bmatrix} = \begin{bmatrix} -[\hat{\boldsymbol{\omega}} \times] & -\mathbf{I}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \end{bmatrix} \cdot \begin{bmatrix} \delta \boldsymbol{\theta} \\ \Delta \mathbf{b} \end{bmatrix} + \begin{bmatrix} -\mathbf{I}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{I}_{3 \times 3} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{n}_r \\ \mathbf{n}_w \end{bmatrix} \quad (158)$$

or

$$\dot{\tilde{\mathbf{x}}} = \mathbf{F}_c \cdot \tilde{\mathbf{x}} + \mathbf{G}_c \cdot \mathbf{n} \quad (159)$$

where

$$\mathbf{F}_c = \begin{bmatrix} -[\hat{\boldsymbol{\omega}} \times] & -\mathbf{I}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \end{bmatrix}, \quad \mathbf{G}_c = \begin{bmatrix} -\mathbf{I}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{I}_{3 \times 3} \end{bmatrix} \quad (160)$$

are the system matrix and the noise matrix, and

$$\tilde{\mathbf{x}} = \begin{bmatrix} \delta \boldsymbol{\theta} \\ \Delta \mathbf{b} \end{bmatrix}, \quad \mathbf{n} = \begin{bmatrix} \mathbf{n}_r \\ \mathbf{n}_w \end{bmatrix} \quad (161)$$

denote the error state and the noise vector, respectively.

As discussed in section 2.1, we assume \mathbf{n}_r and \mathbf{n}_w to be white and independent, so that the continuous time system noise covariance matrix is specified by

$$\mathbf{Q}_c = E [\mathbf{n}(t + \tau) \mathbf{n}^T(t)] = \begin{bmatrix} \mathbf{N}_r & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{N}_w \end{bmatrix} = \begin{bmatrix} \sigma_{r_c}^2 \cdot \mathbf{I}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \sigma_{w_c}^2 \cdot \mathbf{I}_{3 \times 3} \end{bmatrix} \quad (162)$$

2.5 Discrete Time Error State Equations

For implementation of the discrete time Kalman Filter equations, we will need to discretize the above error propagation model. In particular, we will have to find the state transition matrix Φ and the system noise covariance \mathbf{Q}_d [4, Chapter 4.9].

2.5.1 The State Transition Matrix

Since the continuous time system matrix \mathbf{F}_c is constant over the integration time step, we may write the state transition matrix as [4, eq. (2-58a)]

$$\Phi(t + \Delta t, t) = \exp(\mathbf{F}_c \Delta t) \quad (163)$$

$$= \mathbf{I}_{6 \times 6} + \mathbf{F}_c \Delta t + \frac{1}{2!} \mathbf{F}_c^2 \Delta t^2 + \dots \quad (164)$$

Straightforward calculation produces the powers of \mathbf{F}_c as

$$\begin{aligned} \mathbf{F}_c &= \begin{bmatrix} -[\hat{\omega} \times] & -\mathbf{I}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \end{bmatrix}, & \mathbf{F}_c^2 &= \begin{bmatrix} [\hat{\omega} \times]^2 & [\hat{\omega} \times] \\ \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \end{bmatrix} \\ \mathbf{F}_c^3 &= \begin{bmatrix} -[\hat{\omega} \times]^3 & -[\hat{\omega} \times]^2 \\ \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \end{bmatrix}, & \mathbf{F}_c^4 &= \begin{bmatrix} [\hat{\omega} \times]^4 & [\hat{\omega} \times]^3 \\ \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \end{bmatrix} \end{aligned} \quad (165)$$

In keeping with the notation of Lefferts *et al.* [3], we see that the transition matrix has the following block structure

$$\Phi(t + \Delta t, t) = \begin{bmatrix} \Theta & \Psi \\ \mathbf{0}_{3 \times 3} & \mathbf{I}_{3 \times 3} \end{bmatrix} \quad (166)$$

The matrix Θ can be written as

$$\Theta = \mathbf{I}_{3 \times 3} - [\hat{\omega} \times] \Delta t + \frac{1}{2!} [\hat{\omega} \times]^2 \Delta t^2 - \frac{1}{3!} [\hat{\omega} \times]^3 \Delta t^3 + \dots \quad (167)$$

Using the properties of the skew-symmetric matrix $[\hat{\omega} \times]$ (cf. section 1.3.1) and reordering the terms yields

$$\Theta = \mathbf{I}_{3 \times 3} + \left(-\Delta t + \frac{1}{3!} |\hat{\omega}|^2 \Delta t^3 - \dots \right) [\hat{\omega} \times] + \left(\frac{1}{2!} \Delta t^2 - \frac{1}{4!} |\hat{\omega}|^2 \Delta t^4 + \dots \right) [\hat{\omega} \times]^2 \quad (168)$$

$$= \mathbf{I}_{3 \times 3} - \frac{1}{|\hat{\omega}|} \left(|\hat{\omega}| \Delta t - \frac{1}{3!} |\hat{\omega}|^3 \Delta t^3 + \dots \right) [\hat{\omega} \times] + \frac{1}{|\hat{\omega}|^2} \left(1 - \left(1 - \frac{1}{2!} |\hat{\omega}|^2 \Delta t^2 + \frac{1}{4!} |\hat{\omega}|^4 \Delta t^4 - \dots \right) \right) [\hat{\omega} \times]^2 \quad (169)$$

$$= \mathbf{I}_{3 \times 3} - \frac{1}{|\hat{\omega}|} \sin(|\hat{\omega}| \Delta t) [\hat{\omega} \times] + \frac{1}{|\hat{\omega}|^2} (1 - \cos(|\hat{\omega}| \Delta t)) [\hat{\omega} \times]^2 \quad (170)$$

Further expansion of $[\hat{\omega} \times]$ gives

$$\Theta = \cos(|\hat{\omega}| \Delta t) \cdot \mathbf{I}_{3 \times 3} - \sin(|\hat{\omega}| \Delta t) \cdot \left[\frac{\hat{\omega}}{|\hat{\omega}|} \times \right] + (1 - \cos(|\hat{\omega}| \Delta t)) \cdot \frac{\hat{\omega}}{|\hat{\omega}|} \frac{\hat{\omega}}{|\hat{\omega}|}^T \quad (171)$$

where comparison with section 1.4 reveals that Θ is in fact a rotational matrix with ω as the axis of rotation and $|\hat{\omega}| \Delta t$ the corresponding angle.

For small values of $|\hat{\omega}|$, either of the above expressions will lead to numerical instability. By taking the limit and applying L'Hôpital's rule, we arrive at

$$\lim_{|\hat{\omega}| \rightarrow 0} \Theta = \mathbf{I}_{3 \times 3} - \Delta t [\hat{\omega} \times] + \frac{\Delta t^2}{2} [\hat{\omega} \times]^2 \quad (172)$$

Proceeding in similar fashion with the matrix Ψ , we find

$$\Psi = -\mathbf{I}_{3 \times 3} \Delta t + \frac{1}{2!} \Delta t^2 [\hat{\omega} \times] - \frac{1}{3!} \Delta t^3 [\hat{\omega} \times]^2 + \dots \quad (173)$$

$$= -\mathbf{I}_{3 \times 3} \Delta t + \left(\frac{1}{2!} \Delta t^2 - \frac{1}{4!} |\hat{\omega}|^2 \Delta t^4 + \dots \right) [\hat{\omega} \times] + \left(-\frac{1}{3!} \Delta t^3 + \frac{1}{5!} |\hat{\omega}|^2 \Delta t^5 - \dots \right) [\hat{\omega} \times]^2 \quad (174)$$

$$= -\mathbf{I}_{3 \times 3} \Delta t + \frac{1}{|\hat{\omega}|^2} \left(1 - \left(1 - \frac{1}{2!} |\hat{\omega}|^2 \Delta t^2 - \frac{1}{4!} |\hat{\omega}|^4 \Delta t^4 + \dots \right) \right) [\hat{\omega} \times] \quad (175)$$

$$\begin{aligned} &+ \left(-|\hat{\omega}| \Delta t + \left(|\hat{\omega}| \Delta t - \frac{1}{3!} \Delta t^3 + \frac{1}{5!} |\hat{\omega}|^2 \Delta t^5 - \dots \right) \right) [\hat{\omega} \times]^2 \\ &= -\mathbf{I}_{3 \times 3} \Delta t + \frac{1}{|\hat{\omega}|^2} (1 - \cos(|\hat{\omega}| \Delta t)) [\hat{\omega} \times] - \frac{1}{|\hat{\omega}|^3} (|\hat{\omega}| \Delta t - \sin(|\hat{\omega}| \Delta t)) [\hat{\omega} \times]^2 \end{aligned} \quad (176)$$

For small values of $|\hat{\omega}|$, we can again take the limit and apply L'Hôpital's rule and obtain

$$\lim_{|\hat{\omega}| \rightarrow 0} \Psi = -\mathbf{I}_{3 \times 3} \Delta t + \lim_{|\hat{\omega}| \rightarrow 0} \frac{\sin(|\hat{\omega}| \Delta t) \Delta t}{2|\hat{\omega}|} [\hat{\omega} \times] - \lim_{|\hat{\omega}| \rightarrow 0} \frac{(1 - \cos(|\hat{\omega}|)) \Delta t}{3|\hat{\omega}|^2} [\hat{\omega} \times]^2 \quad (177)$$

$$= -\mathbf{I}_{3 \times 3} \Delta t + \lim_{|\hat{\omega}| \rightarrow 0} \frac{\cos(|\hat{\omega}| \Delta t) \Delta t^2}{2} [\hat{\omega} \times] - \lim_{|\hat{\omega}| \rightarrow 0} \frac{\sin(|\hat{\omega}|) \Delta t^2}{6|\hat{\omega}|} [\hat{\omega} \times]^2 \quad (178)$$

$$= -\mathbf{I}_{3 \times 3} \Delta t + \frac{\Delta t^2}{2} [\hat{\omega} \times] - \lim_{|\hat{\omega}| \rightarrow 0} \frac{\cos(|\hat{\omega}|) \Delta t^3}{6} [\hat{\omega} \times]^2 \quad (179)$$

$$= -\mathbf{I}_{3 \times 3} \Delta t + \frac{\Delta t^2}{2} [\hat{\omega} \times] - \frac{\Delta t^3}{6} [\hat{\omega} \times]^2 \quad (180)$$

A note regarding the small ω approximation The error committed by using the small angle approximation is roughly in the order of $\Delta t^{n+2} |\hat{\omega}|^2$, if the non-approximated term is divided by $|\hat{\omega}|^n$.

Therefore, at a sampling rate of 100 Hz, the error for Θ above would be in the order of $10^{-6} \cdot |\hat{\omega}_{\text{thresh}}|^2$, where $|\hat{\omega}_{\text{thresh}}|$ denotes the threshold for the small ω approximation.

2.5.2 The Noise Covariance Matrix \mathbf{Q}_d

The covariance of the noise in the discrete time system can be computed according to [4, p. 171]

$$\mathbf{Q}_d = \int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, \tau) \mathbf{G}_c(\tau) \mathbf{Q}_c \mathbf{G}_c^T(\tau) \Phi^T(t_{k+1}, \tau) d\tau \quad (181)$$

$$= \int_{t_k}^{t_{k+1}} \begin{bmatrix} \Theta & \Psi \\ \mathbf{0}_{3 \times 3} & \mathbf{I}_{3 \times 3} \end{bmatrix} \begin{bmatrix} -\mathbf{I}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{I}_{3 \times 3} \end{bmatrix} \mathbf{Q}_c \begin{bmatrix} -\mathbf{I}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{I}_{3 \times 3} \end{bmatrix} \begin{bmatrix} \Theta^T & \mathbf{0}_{3 \times 3} \\ \Psi^T & \mathbf{I}_{3 \times 3} \end{bmatrix} d\tau \quad (182)$$

$$= \int_{t_k}^{t_{k+1}} \begin{bmatrix} -\Theta & \Psi \\ \mathbf{0}_{3 \times 3} & \mathbf{I}_{3 \times 3} \end{bmatrix} \mathbf{Q}_c \begin{bmatrix} -\Theta^T & \mathbf{0}_{3 \times 3} \\ \Psi^T & \mathbf{I}_{3 \times 3} \end{bmatrix} d\tau \quad (183)$$

$$(184)$$

Assuming the noise to be white and independent (cf. eq. (162)), we can further expand to

$$\mathbf{Q}_d = \int_{t_k}^{t_{k+1}} \begin{bmatrix} -\Theta & \Psi \\ \mathbf{0}_{3 \times 3} & \mathbf{I}_{3 \times 3} \end{bmatrix} \begin{bmatrix} \sigma_r^2 \cdot \mathbf{I}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \sigma_w^2 \cdot \mathbf{I}_{3 \times 3} \end{bmatrix} \begin{bmatrix} -\Theta^T & \mathbf{0}_{3 \times 3} \\ \Psi^T & \mathbf{I}_{3 \times 3} \end{bmatrix} d\tau \quad (185)$$

$$= \int_{t_k}^{t_{k+1}} \begin{bmatrix} \sigma_r^2 \cdot \mathbf{I}_{3 \times 3} + \sigma_w^2 \cdot \Psi \Psi^T & \sigma_w^2 \cdot \Psi \\ \sigma_w^2 \cdot \Psi^T & \sigma_w^2 \cdot \mathbf{I}_{3 \times 3} \end{bmatrix} d\tau \quad (186)$$

where the last step follows from the fact, that Θ is a rotational matrix, and therefore $\Theta \Theta^T$ gives the identity matrix.

The resulting matrix \mathbf{Q}_d has the following structure

$$\mathbf{Q}_d = \begin{bmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \mathbf{Q}_{12}^T & \mathbf{Q}_{22} \end{bmatrix} \quad (187)$$

and the elements follow after considerable algebra as

$$\mathbf{Q}_{11} = \sigma_r^2 \Delta t \cdot \mathbf{I}_{3 \times 3} + \sigma_w^2 \cdot \left(\mathbf{I}_{3 \times 3} \frac{\Delta t^3}{3} + \frac{(|\hat{\omega}| \Delta t)^3}{3} + 2 \frac{\sin(|\hat{\omega}| \Delta t) - 2|\hat{\omega}| \Delta t}{|\hat{\omega}|^5} \cdot [\hat{\omega} \times]^2 \right) \quad (188)$$

$$\mathbf{Q}_{12} = -\sigma_w^2 \cdot \left(\mathbf{I}_{3 \times 3} \frac{\Delta t^2}{2} - \frac{|\hat{\omega}| \Delta t - \sin(|\hat{\omega}| \Delta t)}{|\hat{\omega}|^3} \cdot [\hat{\omega} \times] + \frac{(|\hat{\omega}| \Delta t)^2}{2} + \frac{\cos(|\hat{\omega}| \Delta t) - 1}{|\hat{\omega}|^4} \cdot [\hat{\omega} \times]^2 \right) \quad (189)$$

$$\mathbf{Q}_{22} = \sigma_w^2 \Delta t \cdot \mathbf{I}_{3 \times 3} \quad (190)$$

Similar to the transition matrix, we can derive the form for small $|\hat{\omega}|$ by taking the limit and applying L'Hôpital's rule

$$\lim_{|\hat{\omega}| \rightarrow 0} \mathbf{Q}_{11} = \sigma_r^2 \Delta t \cdot \mathbf{I}_{3 \times 3} + \sigma_w^2 \left(\mathbf{I}_{3 \times 3} \frac{\Delta t^3}{3} + \frac{2\Delta t^5}{5!} \cdot [\hat{\omega} \times]^2 \right) \quad (191)$$

$$\lim_{|\hat{\omega}| \rightarrow 0} \mathbf{Q}_{12} = -\sigma_w^2 \cdot \left(\mathbf{I}_{3 \times 3} \frac{\Delta t^2}{2} - \frac{\Delta t^3}{3!} \cdot [\hat{\omega} \times] + \frac{\Delta t^4}{4!} \cdot [\hat{\omega} \times]^2 \right) \quad (192)$$

2.6 Propagation Equations

Having defined the propagation of the quaternion (cf. section 1.6), and the discrete time state transition- and noise covariance matrices (cf. section 2.5), we can now write the Kalman Filter propagation equations.

Assume we receive gyro measurements ω_{m_k} and $\omega_{m_{k+1}}$, and we have an estimate of the quaternion $\hat{q}_{k|k}$ and the bias $\hat{\mathbf{b}}_{k|k}$ at timestep k , together with the corresponding covariance matrix $\mathbf{P}_{k|k}$. From this, using eq. (136), we also have an estimate of $\hat{\omega}_{k|k}$.

We now proceed as follows:

1. We propagate the bias using the discretized form of eq. (135) as

$$\hat{\mathbf{b}}_{k+1|k} = \hat{\mathbf{b}}_{k|k} \quad (193)$$

2. Using the measurement $\omega_{m_{k+1}}$ and $\hat{\mathbf{b}}_{k+1|k}$, we form the estimate of the new turn rate according to eq. (136) as

$$\hat{\omega}_{k+1|k} = \omega_{m_{k+1}} - \hat{\mathbf{b}}_{k+1|k} \quad (194)$$

3. We propagate the quaternion using a first order integrator (cf. section 1.6.2) with $\hat{\omega}_{k|k}$ and $\hat{\omega}_{k+1|k}$ to obtain $\hat{q}_{k+1|k}$.
4. From the formulas in sections 2.5.1 and 2.5.2 we compute the state transition matrix Φ and the discrete time noise covariance matrix \mathbf{Q}_d .
5. We compute the state covariance matrix according to the Extended Kalman Filter equation

$$\mathbf{P}_{k+1|k} = \Phi \mathbf{P}_{k|k} \Phi^T + \mathbf{Q}_d \quad (195)$$

3 Update

In the update phase of the Kalman filter, we will use an exteroceptive sensor to measure orientation. Common examples are star- and sunsensors. We will now consider a simplified model of a sunsensor, a more detailed version of which is presented in [7].

3.1 Measurement Models

In order to provide exteroceptive information to the attitude filter, we can envisage a multitude of sensors. Each is characterized by its particular sensor model.

In the following, we will present a selection of sensors models, followed by the general procedure to fuse these measurements with the filter estimate.

3.1.1 Sun Sensor

In this model, we assume that we are able to measure a projection of the vector from the sensor to the sun with respect to sensor frame ${}^S\mathbf{r}_\odot$, whose representation in the global coordinate frame ${}^G\mathbf{r}_\odot$ is known.

The relationship between the two is given by the rotation

$${}^S\mathbf{r}_\odot = {}^S\mathbf{C} {}^G\mathbf{r}_\odot \quad (196)$$

The rotational matrix ${}^S\mathbf{C}$ can be decomposed as

$${}^S\mathbf{C} = {}^S\mathbf{C} {}^L\mathbf{C} \quad (197)$$

where the transformation ${}^S\mathbf{C}$ between sensor frame $\{S\}$ and spacecraft frame $\{L\}$ is known and fixed, and the transformation ${}^L\mathbf{C}$ is a function of the attitude quaternion.

The actual measurement \mathbf{z} will be a projection of the vector \mathbf{r}_\odot , corrupted by zero-mean, white, Gaussian noise \mathbf{n}_m

$$\mathbf{z} = \mathbf{\Pi} {}^S\mathbf{C} {}^L\mathbf{C} {}^G\mathbf{r}_\odot + \mathbf{n}_m \quad (198)$$

where $\mathbf{\Pi}$ is the projection matrix.

The measurement noise is characterized by

$$E[\mathbf{n}_m] = \mathbf{0} \quad (199)$$

$$E[\mathbf{n}_m \mathbf{n}_m^T] = \mathbf{R} \quad (200)$$

For the update phase of the Kalman filter, we need to relate the measurement error $\tilde{\mathbf{z}}$ to the state vector.

$$\tilde{\mathbf{z}} = \mathbf{z} - \hat{\mathbf{z}} = \mathbf{\Pi} {}^S\mathbf{C} \left({}^L\mathbf{C}(\bar{q}) - \hat{L}\mathbf{C}(\hat{q}) \right) \cdot {}^G\mathbf{r}_\odot + \mathbf{n}_m \quad (201)$$

From the error model (cf. section 2.3) and the properties of the rotational matrix (cf. section 1.4) we recall the following expressions

$$\bar{q} = \delta\bar{q} \otimes \hat{q} \quad (202)$$

$$\begin{aligned} {}^L\mathbf{C}(\bar{q}) &= {}^L\mathbf{C}(\delta\bar{q} \otimes \hat{q}) \\ &= {}^L\mathbf{C}(\delta\bar{q}) \cdot \hat{L}\mathbf{C}(\hat{q}) \end{aligned} \quad (203)$$

$${}^L\mathbf{C}(\delta\bar{q}) \approx \mathbf{I} - [\delta\boldsymbol{\theta} \times] \quad (204)$$

and hence ¹

$$\frac{L}{G}\mathbf{C}(\bar{q}) - \frac{L}{G}\mathbf{C}(\hat{q}) = \left(\frac{L}{L}\mathbf{C}(\delta\bar{q}) - \mathbf{I}_{3 \times 3} \right) \cdot \frac{L}{G}\mathbf{C}(\hat{q}) = -[\delta\boldsymbol{\theta} \times] \cdot \frac{L}{G}\mathbf{C}(\hat{q}) \quad (206)$$

We can now write

$$\tilde{\mathbf{z}} = \boldsymbol{\Pi}_L^S \mathbf{C} \left(\frac{L}{L}\mathbf{C}(\delta\bar{q}) - \mathbf{I} \right) \frac{L}{G}\mathbf{C}(\hat{q}) \cdot {}^G\mathbf{r}_{\odot} + \mathbf{n}_m \quad (207)$$

$$\approx \boldsymbol{\Pi}_L^S \mathbf{C} (-[\delta\boldsymbol{\theta} \times]) \frac{L}{G}\mathbf{C}(\hat{q}) \cdot {}^G\mathbf{r}_{\odot} + \mathbf{n}_m \quad (208)$$

$$= \boldsymbol{\Pi}_L^S \mathbf{C} \left[\frac{L}{G}\mathbf{C}(\hat{q}) {}^G\mathbf{r}_{\odot} \times \right] \cdot \delta\boldsymbol{\theta} + \mathbf{n}_m \quad (209)$$

$$= \left[\boldsymbol{\Pi}_L^S \mathbf{C} \left[\frac{L}{G}\mathbf{C}(\hat{q}) {}^G\mathbf{r}_{\odot} \times \right] \quad \mathbf{0} \right] \cdot \begin{bmatrix} \delta\boldsymbol{\theta} \\ \tilde{\mathbf{b}} \end{bmatrix} + \mathbf{n}_m \quad (210)$$

so that the measurement matrix \mathbf{H} corresponds to

$$\mathbf{H} = \left[\boldsymbol{\Pi}_L^S \mathbf{C} \left[\frac{L}{G}\mathbf{C}(\hat{q}) {}^G\mathbf{r}_{\odot} \times \right] \quad \mathbf{0} \right] \quad (211)$$

3.2 Kalman Filter Update

Given the propagated state estimates $\hat{q}_{k+1|k}$ and $\hat{\mathbf{b}}_{k+1|k}$, as well as their covariance matrix $\mathbf{P}_{k+1|k}$, the current measurement $\mathbf{z}(k+1)$, and the measurement matrix \mathbf{H} , we can update our estimate in the following way:

1. Compute the measurement matrix $\mathbf{H}(k)$ according to eq. (211)
2. Compute residual \mathbf{r} according to

$$\mathbf{r} = \mathbf{z} - \hat{\mathbf{z}} \quad (212)$$

$$(213)$$

3. Compute the covariance of the residual \mathbf{S} as

$$\mathbf{S} = \mathbf{H}\mathbf{P}\mathbf{H}^T + \mathbf{R} \quad (214)$$

4. Compute the Kalman gain \mathbf{K}

$$\mathbf{K} = \mathbf{P}\mathbf{H}^T\mathbf{S}^{-1} \quad (215)$$

5. Compute the correction $\Delta\hat{\mathbf{x}}(+)$

$$\Delta\hat{\mathbf{x}}(+) = \begin{bmatrix} \delta\hat{\boldsymbol{\theta}}(+) \\ \Delta\hat{\mathbf{b}}(+) \end{bmatrix} = \begin{bmatrix} 2 \cdot \delta\hat{\mathbf{q}}(+) \\ \Delta\hat{\mathbf{b}}(+) \end{bmatrix} = \mathbf{K}\mathbf{r} \quad (216)$$

6. Update the quaternion according to

$$\delta\hat{q} = \begin{bmatrix} \delta\hat{\mathbf{q}}(+) \\ \sqrt{1 - \delta\hat{\mathbf{q}}(+)^T \delta\hat{\mathbf{q}}(+)} \end{bmatrix} \quad (217)$$

or, if $\delta\hat{\mathbf{q}}(+)^T \delta\hat{\mathbf{q}}(+) > 1$, using

$$\delta\hat{q} = \frac{1}{\sqrt{1 + \delta\hat{\mathbf{q}}(+)^T \delta\hat{\mathbf{q}}(+)}} \cdot \begin{bmatrix} \delta\hat{\mathbf{q}}(+) \\ 1 \end{bmatrix} \quad (218)$$

$$\hat{q}_{k+1|k+1} = \delta\hat{q} \otimes \hat{q}_{k+1|k} \quad (219)$$

¹Note here the difference to the expression obtained if we were using an *additive* instead of *multiplicative* error model, where $\bar{q} = \hat{q} + \Delta\bar{q}$ with $\Delta\bar{q} = [\Delta\mathbf{q}^T \quad \Delta q_4]^T$:

$$\frac{L}{G}\mathbf{C}(\hat{q} + \Delta\bar{q}) - \frac{L}{G}\mathbf{C}(\hat{q}) = 4\hat{q}_4 \Delta q_4 \mathbf{I}_{3 \times 3} - 2\hat{q}_4 [\Delta\mathbf{q} \times] - 2\Delta q_4 [\hat{\mathbf{q}} \times] + 2\Delta\mathbf{q}\hat{\mathbf{q}}^T + 2\hat{\mathbf{q}}\Delta\mathbf{q}^T \quad (205)$$

7. Update the bias

$$\hat{\mathbf{b}}_{k+1|k+1} = \hat{\mathbf{b}}_{k+1|k} + \Delta\hat{\mathbf{b}}(+)$$
 (220)

8. Update the estimated turn rate using the new estimate for the bias

$$\hat{\omega}_{k+1|k+1} = \omega_{m_{k+1}} - \hat{\mathbf{b}}_{k+1|k+1}$$
 (221)

9. Compute the new updated Covariance matrix

$$\mathbf{P}_{k+1|k+1} = (\mathbf{I}_{6 \times 6} - \mathbf{KH})\mathbf{P}_{k+1|k}(\mathbf{I}_{6 \times 6} - \mathbf{KH})^T + \mathbf{KRK}^T$$
 (222)

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