Jacobian for conversion from Euler Angles to Quaternions

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This Techreport derives the Jacobian needed for error conversion when changing from Euler angles to quaternion based attitude representation. When applying the following results, it is important to pay careful attention to the super- and subscripts, i.e., the convention used in this report to express the global frame with respect to the local frame.

1 Rotational Matrix and Quaternions

As shown in [2], the rotational matrix ${}^{L}_{G}\mathbf{C}$ can be expressed in terms of the corresponding quaternion $\bar{q} = \begin{bmatrix} \mathbf{q} & q_4 \end{bmatrix}^T$ as

$$ {}^{L}_{G}\mathbf{C}(\bar{q}) = \left(2q_4^2 - 1\right)\mathbf{I}_{3\times 3} - 2q_4\lfloor \mathbf{q}\times\rfloor + 2\mathbf{q}\mathbf{q}^{\mathrm{T}} \tag{1} $$

where $\lfloor \mathbf{q}\times\rfloor$ denotes the skew-symmetric cross-product matrix

$$ \lfloor \mathbf{q}\times\rfloor = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix} \tag{2} $$

Note the following property of a skew-symmetric matrix

$$ \mathbf{C}\lfloor \mathbf{a}\times\rfloor\mathbf{C}^T = \lfloor \mathbf{C}\mathbf{a}\times\rfloor \tag{3} $$

where $\mathbf{a}$ is a vector and $\mathbf{C}$ a rotational matrix.

In case of only a very small rotation $\delta\bar{q}$, we can use the small angle approximation to simplify Eq. (1). We can write the quaternion describing a small rotation as

$$ \delta\bar{q} = \begin{bmatrix} \delta\mathbf{q} \\ \delta q_4 \end{bmatrix} \tag{4} $$

$$ = \begin{bmatrix} \hat{\mathbf{k}}\sin(\delta\theta/2) \\ \cos(\delta\theta/2) \end{bmatrix} \tag{5} $$

$$ \approx \begin{bmatrix} \frac{1}{2}\delta\boldsymbol{\theta} \\ 1 \end{bmatrix} \tag{6} $$

leading to the following expression for the corresponding rotational matrix

$$ {}^{L}_{G}\mathbf{C}(\delta\bar{q}) \approx \mathbf{I}_{3\times 3} - \lfloor \delta\boldsymbol{\theta}\times\rfloor \tag{7} $$

Note that $\delta\boldsymbol{\theta}$ is the product of the infinitesimal rotation angle $\delta\theta$ and the axis of rotation $\hat{\mathbf{k}}$.

Using a *multiplicative* error model for the quaternion, we can decompose the true orientation into a quaternion product of error quaternion $\delta\bar{q}$ and estimated quaternion ${}^{\hat{L}}_{G}\hat{\bar{q}}$

$$ {}^{L}_{G}\bar{q} = {}^{L}_{\hat{L}}\delta\bar{q} \otimes {}^{\hat{L}}_{G}\hat{\bar{q}} \tag{8} $$

As a consequence of the chosen quaternion convention, this entails

$$
\begin{aligned}
{}^{L}_{G}\mathbf{C}(\bar{q}) &= {}^{L}_{G}\mathbf{C}(\delta\bar{q} \otimes \hat{\bar{q}}) && (9)\\
&= {}^{L}_{\hat{L}}\mathbf{C}(\delta\bar{q}) \cdot {}^{\hat{L}}_{G}\mathbf{C}(\hat{\bar{q}}) && (10)\\
&= (\mathbf{I}_{3\times 3} - \lfloor \delta\boldsymbol{\theta} \times \rfloor) \cdot {}^{\hat{L}}_{G}\mathbf{C}(\hat{\bar{q}}) && (11)
\end{aligned}
$$

2 Rotational Matrix and Euler Angles

The rotational matrix can be parametrized by a three angles (the so-called Euler angles) describing a sequence of rotations.

In its most general form, this can be written as

$$
{}^{L}_{G}\mathbf{C}(\alpha, \beta, \gamma) = \mathbf{C}(\alpha, \hat{\mathbf{i}}) \cdot \mathbf{C}(\beta, \hat{\mathbf{j}}) \cdot \mathbf{C}(\gamma, \hat{\mathbf{k}}) \tag{12}
$$

where α, β, γ are the angles of rotation and $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ denote the unit vectors along the corresponding axes of rotation. For a more detailed overview, please refer to [1].

Employing Eqs. (3) and (11), and neglecting second order terms, we can decompose Eq. (12) in a product of error term and estimated rotational matrix as

$$
\begin{aligned}
{}^{L}_{G}\mathbf{C}(\alpha, \beta, \gamma) &= (\mathbf{I}_{3\times 3} - \lfloor \delta\alpha\hat{\mathbf{i}} \times \rfloor) \cdot \mathbf{C}(\hat{\alpha}, \hat{\mathbf{i}}) \cdot (\mathbf{I}_{3\times 3} - \lfloor \delta\beta\hat{\mathbf{j}} \times \rfloor) \cdot \mathbf{C}(\hat{\beta}, \hat{\mathbf{j}}) \cdot (\mathbf{I}_{3\times 3} - \lfloor \delta\gamma\hat{\mathbf{k}} \times \rfloor \cdot \mathbf{C}(\hat{\gamma}, \hat{\mathbf{k}}) && (13)\\
&= (\mathbf{I}_{3\times 3} - \lfloor \delta\alpha\hat{\mathbf{i}} \times \rfloor) \cdot \left(\mathbf{I}_{3\times 3} - \mathbf{C}(\hat{\alpha}, \hat{\mathbf{i}}) \lfloor \delta\beta\hat{\mathbf{j}} \times \rfloor \mathbf{C}(\hat{\alpha}, \hat{\mathbf{i}})^{T}\right) \cdot \mathbf{C}(\hat{\alpha}, \hat{\mathbf{i}}) \cdot \mathbf{C}(\hat{\beta}, \hat{\mathbf{j}})\\
&\quad \cdot (\mathbf{I}_{3\times 3} - \lfloor \delta\gamma\hat{\mathbf{k}} \times \rfloor \cdot \mathbf{C}(\hat{\gamma}, \hat{\mathbf{k}}) && (14)\\
&\simeq (\mathbf{I}_{3\times 3} - \lfloor \delta\alpha\hat{\mathbf{i}} \times \rfloor - \lfloor \mathbf{C}(\hat{\alpha}, \hat{\mathbf{i}}) \delta\beta\hat{\mathbf{j}} \times \rfloor) \cdot \left(\mathbf{I}_{3\times 3} - \mathbf{C}(\hat{\alpha}, \hat{\mathbf{i}}) \mathbf{C}(\hat{\beta}, \hat{\mathbf{j}}) \lfloor \delta\gamma\hat{\mathbf{k}} \times \rfloor \mathbf{C}(\hat{\beta}, \hat{\mathbf{j}})^{T} \mathbf{C}(\hat{\alpha}, \hat{\mathbf{i}})^{T}\right)\\
&\quad \cdot \mathbf{C}(\hat{\alpha}, \hat{\mathbf{i}}) \cdot \mathbf{C}(\hat{\beta}, \hat{\mathbf{j}}) \cdot \mathbf{C}(\hat{\gamma}, \hat{\mathbf{k}}) && (15)\\
&\simeq \left(\mathbf{I}_{3\times 3} - \lfloor \delta\alpha\hat{\mathbf{i}} \times \rfloor - \lfloor \mathbf{C}(\hat{\alpha}, \hat{\mathbf{i}}) \delta\beta\hat{\mathbf{j}} \times \rfloor - \lfloor \mathbf{C}(\hat{\alpha}, \hat{\mathbf{i}}) \cdot \mathbf{C}(\hat{\beta}, \hat{\mathbf{j}}) \delta\gamma\hat{\mathbf{k}} \times \rfloor\right) \cdot \mathbf{C}(\hat{\alpha}, \hat{\mathbf{i}}) \cdot \mathbf{C}(\hat{\beta}, \hat{\mathbf{j}}) \cdot \mathbf{C}(\hat{\gamma}, \hat{\mathbf{k}})\\
& && (16)\\
&= \left(\mathbf{I}_{3\times 3} - \lfloor \delta\alpha\hat{\mathbf{i}} - \mathbf{C}(\hat{\alpha}, \hat{\mathbf{i}}) \delta\beta\hat{\mathbf{j}} - \mathbf{C}(\hat{\alpha}, \hat{\mathbf{i}}) \cdot \mathbf{C}(\hat{\beta}, \hat{\mathbf{j}}) \delta\gamma\hat{\mathbf{k}} \times \rfloor\right) \cdot \mathbf{C}(\hat{\alpha}, \hat{\mathbf{i}}) \cdot \mathbf{C}(\hat{\beta}, \hat{\mathbf{j}}) \cdot \mathbf{C}(\hat{\gamma}, \hat{\mathbf{k}}) && (17)
\end{aligned}
$$

Comparison with Eq. (11) reveals that

$$
\begin{aligned}
\delta\boldsymbol{\theta} &= \delta\alpha\hat{\mathbf{i}} + \mathbf{C}(\hat{\alpha}, \hat{\mathbf{i}}) \delta\beta\hat{\mathbf{j}} + \mathbf{C}(\hat{\alpha}, \hat{\mathbf{i}}) \cdot \mathbf{C}(\hat{\beta}, \hat{\mathbf{j}}) \delta\gamma\hat{\mathbf{k}} && (18)\\
&= \mathbf{H} \begin{bmatrix} \delta\alpha \\ \delta\beta \\ \delta\gamma \end{bmatrix} && (19)
\end{aligned}
$$

where

$$
\mathbf{H} = \begin{bmatrix} \hat{\mathbf{i}} & \mathbf{C}(\hat{\alpha}, \hat{\mathbf{i}}) \hat{\mathbf{j}} & \mathbf{C}(\hat{\alpha}, \hat{\mathbf{i}}) \mathbf{C}(\hat{\beta}, \hat{\mathbf{j}}) \hat{\mathbf{k}} \end{bmatrix} \tag{20}
$$

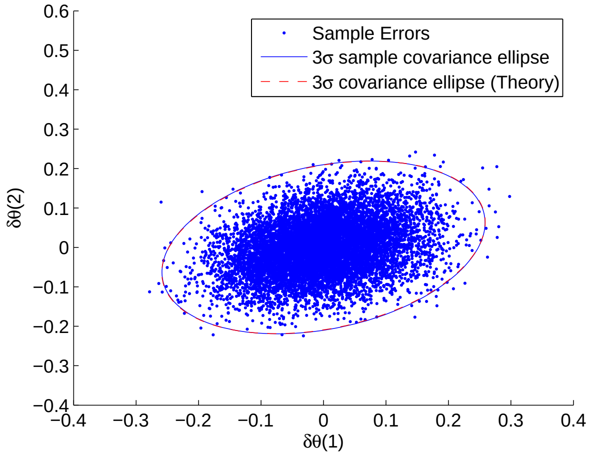
is the Jacobian.

The covariance can then be converted as

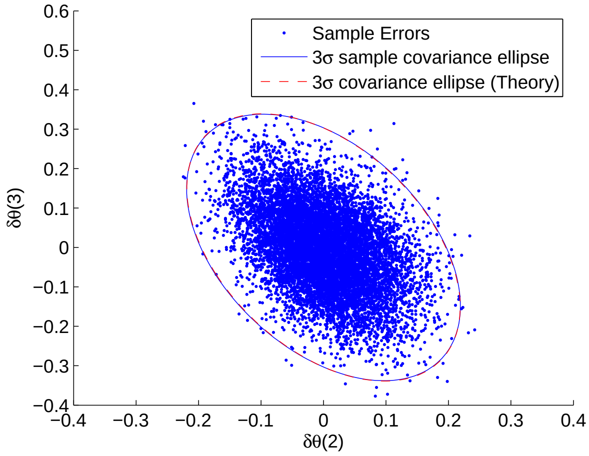
$$
\mathbf{P}_{\delta\boldsymbol{\theta}} = \mathbf{H} \mathbf{P}_{\delta\alpha, \delta\beta, \delta\gamma} \mathbf{H}^{T} \tag{21}
$$

3 Monte Carlo Simulation

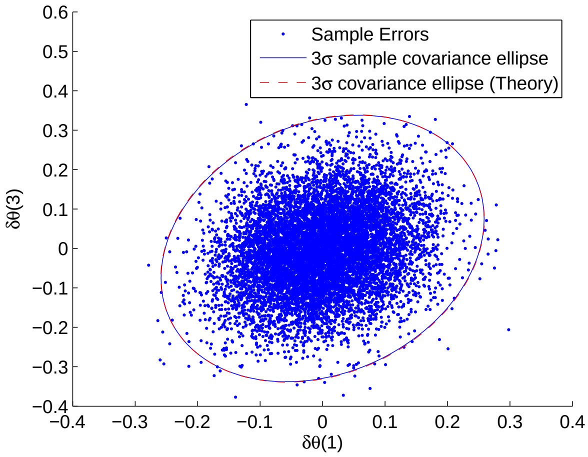
The results for covariance transformation have been verified through Monte Carlo simulations. Fig. 3 shows exemplary results for the parameters given in Tab. 1. The results show good correspondence between the covariance computed by Eq. (21) and the sample covariance.



(a)



(b)



(c)

Figure 1: Monte Carlo error distribution. Overlay of sample covariance and theoretically determined covariance.

Quantity	Value
No. of samples	10000
$\hat{\mathbf{i}}$	$[\,0 \quad 0 \quad 1\,]^T$
$\hat{\mathbf{j}}$	$[\,0 \quad 1 \quad 0\,]^T$
$\hat{\mathbf{k}}$	$[\,1 \quad 0 \quad 0\,]^T$
α	$\pi/3$
β	$\pi/4$
γ	$-\pi/5$
$\mathbf{P}_{\delta\alpha,\delta\beta,\delta\gamma}$	$.0072 \text{ rad}^2 \cdot \mathbf{I}_{3\times 3}$

Table 1: Monte Carlo Parameters

References

[1] J. J. Craig. *Introduction to robotics: mechanics and control.* Pearson Prentice Hall, Upper Saddle River, New Jersey, 3 edition, 2005.

[2] N. Trawny and S. I. Roumeliotis. Indirect Kalman filter for 3D attitude estimation. Technical Report 2005-002, University of Minnesota, Dept. of Comp. Sci. & Eng., Jan. 2005.